Cluster algebras and cluster categories via surfaces

Karin Baur

11th June 2026

Contents

Cluster algebras 1
0.1 Introduction 1

1 Surface combinatorics 5
1.1 Marked surfaces 5
1.2 Arcs and triangulations 6
1.3 Flips of arcs in triangulations 9
1.4 The rank of a marked surface 11
1.5 Tagged arcs 13
1.6 The quiver of a triangulation 17
1.7 The cluster algebra of a quiver 21
1.8 The cluster algebra of a triangulation of a marked surface 30

2 Cluster categories 35
2.1 Categorifications of cluster algebras 35
2.2 Cluster categories from acyclic quivers 36
2.3 The cluster category of a marked surface 41

References 49

Cluster algebras

0.1 Introduction

Cluster algebras were introduced by Sergey Fomin and Andrei Zelevinsky around 2000, cf. [52] who were motivated by phenomena observed in the study of total positivity (for algebraic groups) and in the construction of dual canonical bases in Lie theory.

Totally positive matrices were first studied in the 30s. They are $n \times n$-matrices where all minors are positive. One can show that a totally positive matrix has n distinct real eigenvalues. After these results were found in the 50s, there was not much research around the topic until Lusztig generalized the notion to (reductive) algebraic groups in the 90s. One can see that in order to check whether an element of a matrix group is totally positive, one does not need to check all the minors, there are some redundancies or rather, there are some relations between the minors.

The idea in the construction of a cluster algebra is to start with a tuple of n variables

- $\underline{x} = (x_1, \ldots, x_n)$ and consider the field of fractions $\mathbb{Q}(x_1, \ldots, x_n)$ they define.
- Then one defines an exchange rule (mutation rule) to replace elements of the tuple one by one by a rational function in the previous tuple. $x_i \rightsquigarrow x_i' \in \mathbb{Q}(x_1, \ldots, x_n)$. The mutation rule gives $x_i \cdot x_i'$ as a binomial in the variables of $\underline{x}$.
- If $\mathcal{X}$ is the collection of all variables obtained through this, $\mathcal{X} \subset \mathbb{Q}(x_1, \ldots, x_n)$, one defines the cluster algebra $\mathcal{A}(\underline{x}, \text{rule})$ of $\underline{x}$ (with the mutation rule) to be the subalgebra of the field of fraction generated by $\mathcal{X}$.

Examples of cluster algebras are coordinate rings of algebraic varieties, e.g., Bruhat cells, Grassmannians, or Schubert cells, [18, 87, 110], for example.

Since their introduction, connections between cluster algebras and various fields of mathematics have been discovered, including to the following.

- Combinatorics: polyhedra, snake graphs, dimer models, triangulations of surfaces, see [7, 9, 33, 35, 48, 50, 54, 64, 84, 94, 96, 103, 109, 112] for early examples;
- Root systems in Lie theory (Dynkin diagrams), Kac–Moody Lie algebras, quantized enveloping algebras, see for example [18, 19, 38, 41, 49, 58, 76];
- Representation theory of finite-dimensional algebras: cluster categories, cluster-tilted algebras, preprojective algebras, 2-Calabi–Yau categories, see [26, 27, 80, 114];
- Poisson geometry and algebraic geometry: cluster varieties, Grassmannians, flag varieties, scattering diagrams, Poisson structures on Lie groups, see [22, 44, 47, 59, 61, 66, 91, 110];
- Dynamical systems: pentagram map, Y-systems, frieze patterns, see [4, 5, 12, 13, 30, 60, 67, 73, 93];

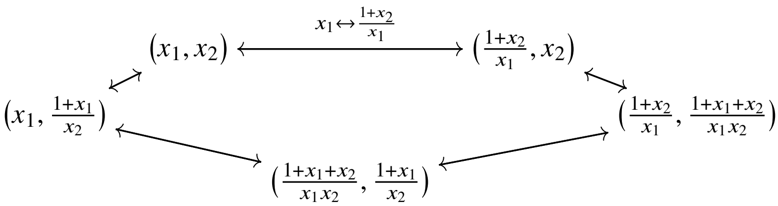


Figure 1. The five clusters of $\mathcal{A}(1,1)$.

- Teichmüller theory: lambda lengths, Penner coordinates, laminations, cluster ensembles, see [46, 50, 62, 75, 86, 113];
- Knot theory: Chern–Simons invariants, Legendrian knots, see [15, 71, 89, 98, 111];
- Mirror symmetry, mathematical physics, Donaldson–Thomas invariants, statistical mechanics. [16, 56, 68, 90, 99, 105].

The **Cluster algebras portal**[1] of Sergey Fomin provides links to activities around cluster algebras.

Example 0.1.1. We take (x_1, x_2) and as a mutation rule, we use a *quiver* (an oriented graph) with two vertices $1, 2$ and one arrow, $Q : 1 \longrightarrow 2$.

The *mutation* μ_i at i (for $i = 1, 2$) is given as follows:

(a) $x_i \cdot x_i' = \prod_{j \to i} x_j + \prod_{j \leftarrow i} x_j$;

(b) (add a short cut for any 2-path $j_1 \longrightarrow i \longrightarrow j_2$: this will only appear later)

(c) invert all arrows at i;

(d) (remove all 2-cycles created through the above).

In the figure, the quiver associated to (x_1, x_2) is $1 \longrightarrow 2$. The two neighboured tuples have the quiver $1 \longleftarrow 2$. The quiver of the right most tuple is $1 \longrightarrow 2$ again and the one of the tuple in the lowest row is $1 \longleftarrow 2$. Then to go from this back to the left most tuple, we have identified $(\frac{1+x_1+x_2}{x_1 x_2}, \frac{1+x_1}{x_2})$ with $(\frac{1+x_1}{x_2}, \frac{1+x_1+x_2}{x_1 x_2})$ in Figure 1. Thus the recurrence gives a 5-periodic sequence. Hence by definition

$$\mathcal{A}(1,1) := \left\langle x_1, x_2, \frac{x_2+1}{x_1}, \frac{x_1+x_2+1}{x_1 x_2}, \frac{x_1+1}{x_2} \right\rangle \subset \mathbb{Q}(x_1, x_2)$$

As this cluster algebra has only finitely many variables, we call it a cluster algebra of *finite type*. We note that all cluster variables are Laurent polynomials in x_1 and x_2 with

[1] https://dept.math.lsa.umich.edu/~fomin/cluster.html

integer coefficients. This is an illustration of the *Laurent phenomenon*. Furthermore, all the coefficients appearing are non-negative, so the cluster variables all belong to $\mathbb{Z}_{\geq 0}[x_1^{\pm 1}, x_2^{\pm 1}]$. This observation is called the *positivity phenomenon*.

This example has been studied as a recurrence given by $x_{m+1} \cdot x_{m-1} = x_m + 1$, called the *pentagramma mirificum*. It has been studied for a long time, e.g., by J. Napier around 1600, or in the 19th century by W. Spence and A. Cayley, see [106] for more details.

Chapter 1

Surface combinatorics

One of the first examples of a cluster algebra arises from triangulations of a convex polygon. Fomin, Shapiro and Thurston associate cluster algebras to arbitrary surfaces as we recall now. Details can be found in [50].

1.1 Marked surfaces

Let S be a connected oriented Riemann surface with boundary ∂S. We allow the boundary to be empty. A surface is *closed* if its boundary ∂S is empty. Let $\emptyset \neq M \subset S$ be a finite set of points of S such that every connected component of ∂S contains at least one point of M.

Definition 1.1.1. With the notations above, the pair (S, M) is called a *marked surface*, or a *surface*. The elements of M are the *marked points*. Marked points in the interior of S are called *punctures*.

This notion of marked surface is also called a *ciliated surface* in the literature, e.g., in the work [45] of Fock and Goncharov.

Cluster structures for surfaces with infinite sets (discrete or continuous) for M have also been defined, see the articles [65, 72, 74] for early examples with discrete sets of marked points.

Remark 1.1.2. Up to homeomorphism, a marked surface (S, M) is determined by

(1) the genus g of S,

(2) the number b of connected components of the boundary of S,

(3) the partitioning of the marked points in ∂S into b parts, giving the number of marked points on each boundary component,

(4) the number p of punctures among the points in M.

See for example Chapter 1 of [42] where references for details can be found.

Figure 1.1 shows several examples of surfaces of genus ≤ 2 with at most two boundary components.

Remark 1.1.3. In the following, we exclude surfaces which do not have any triangulations or which only have one triangulation. These surfaces are the sphere with at most three punctures, the monogon with at most one puncture, the unpunctured digon and the unpunctured triangle.

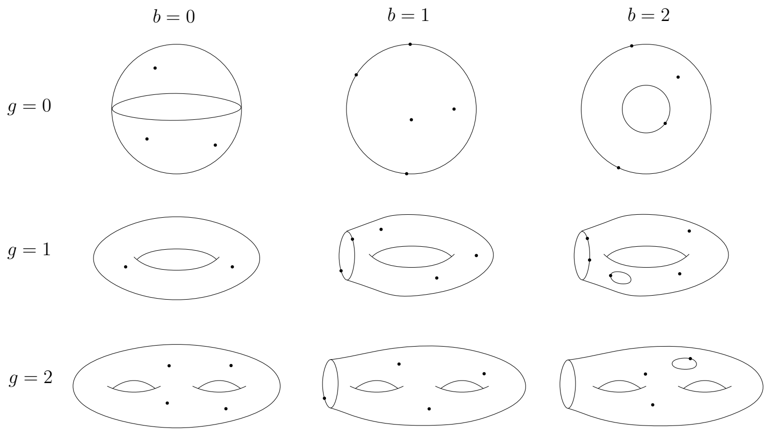


Figure 1.1. Examples with small genus and small number of boundary components.

1.2 Arcs and triangulations

We use arcs in a surface to define cluster variables. The clusters arise from maximal collections of pairwise non-crossing arcs.

Definition 1.2.1. A (simple) *arc* γ in (S, M) is a curve in S, up to isotopy fixing the endpoints of γ, such that

(1) the endpoints of γ are in M,

(2) the interior of γ is disjoint from M and from the boundary ∂S,

(3) the curve γ does not cut out an unpunctured monogon or digon,

(4) the curve γ does not cross itself (its endpoints may coincide).

Curves which satisfy (1), (2) and (3) but have self-crossings are called *generalized arcs*. Curves which lie in ∂S, with endpoints in M but whose interior is disjoint from M are called *boundary segments*. Non-contractible curves in S disjoint from ∂S are called *closed loops*.

One can show that there are only few surfaces with finitely many arcs:

Proposition 1.2.2. *The marked surface (S, M) has finitely many arcs if and only if S is a polygon with at most one puncture.*

Exercise 1. Prove Proposition 1.2.2.

In the construction of a cluster algebra for (S, M), arcs correspond to cluster variables: clusters arise from so-called compatible arcs as we now introduce.

Definition 1.2.3. (1) Two arcs γ, γ' are *compatible* if they do not cross in the interior of S (if there are representatives in their isotopy classes which do not cross).

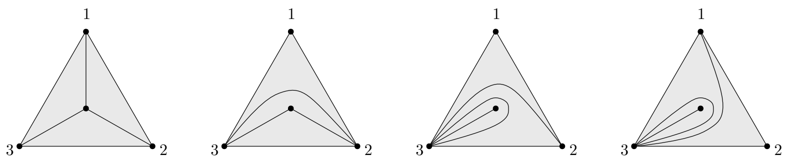


Figure 1.2. Representatives for the ten triangulations of a triangle with one puncture.

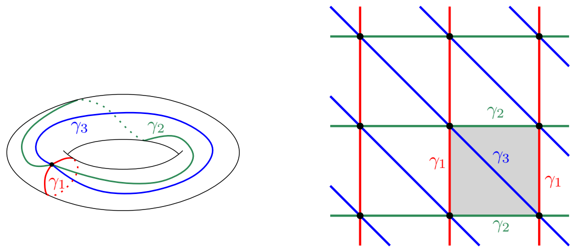


Figure 1.3. A triangulation of the once-punctured torus.

(2) A *triangulation* of a surface (S, M) is a maximal collection or pairwise compatible arcs together with all boundary segments. The arcs of a triangulation cut the surface into regions called *triangles*. If a triangle has only two distinct sides, it is called *self-folded*.

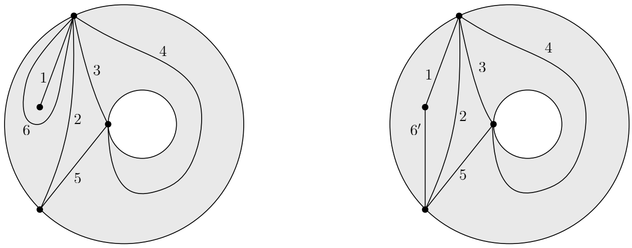
Example 1.2.4. Let (S, M) be a triangle with one puncture. So $M = \{1, 2, 3, p\}$ where $1, 2, 3$ are the three vertices of the triangle and p is the puncture. This marked surface has ten triangulations. They are shown in Figure 1.2 (up to rotations). Two of these triangulations contain a self-folded triangle. The first triangulation in the figure is invariant under rotations by 120 and 240 degrees. Rotating any of the other three pictures by 120 or 240 degrees gives another triangulation of the surface. Figure 1.8 below shows all ten triangulations of the once punctured triangle.

Example 1.2.5. Let (S, M) be a torus with one puncture. Every triangulation of this surface has three arcs. Figure 1.3 shows two versions of a triangulation of (S, M). The one on the right is drawn using the universal cover of the torus.

Example 1.2.6. Two triangulations of an annulus with two marked points on the outer boundary, one marked point on the inner boundary and with one puncture are shown in Figure 1.4.

A consequence of Proposition 1.2.2 is the following:

Corollary 1.2.7. *The marked surface (S, M) has finitely many triangulations if and only if S is a polygon with at most one puncture.*



(a) Triangulation T **(b)** Result after flipping arc 6

Figure 1.4. Two triangulations of an annulus with one puncture.

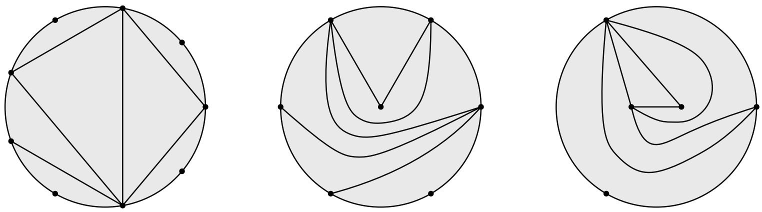


Figure 1.5. Triangulations of different disks, each of them using six arcs.

Notation 1.2.8. Let (S, M) be a marked surface. We write g for the genus of S, b the number of connected components of the boundary of S, p the number of punctures and $c := |M| - p$ for the number of marked points on the boundary.

Example 1.2.9. Let S be a disk with at most two punctures, i.e., $g = 0$, $b = 1$ and $p = |M \setminus \partial S| \leq 2$. Then the marked surface is one of the following:

(1) In case $p = 0$, S is a disk with c marked points on the boundary (or a c-gon), where $c \geq 4$. Any triangulation of S uses exactly $c - 3$ arcs (diagonals).

(2) If $p = 1$, S is a disk with one puncture (with $c \geq 2$) Any triangulation of S is given by c arcs.

(3) In case $p = 2$, we have a twice punctured disk. The triangulations of S contain $c + 3$ arcs.

We see in Example 1.2.9, the number of arcs of a triangulation increases by 3 whenever we add a puncture to the disk. This phenomenon is a general fact, see Lemma 1.4.2 below.

Exercise 2. Let (S, M) be a disk with at most two punctures and c marked points on the boundary. Check that the number of arcs in a triangulation of S is as claimed in Example 1.2.9 (1)-(3).

To illustrate Example 1.2.9, Figure 1.5 shows triangulations of different disks with up to two punctures. All three triangulations have six arcs. A further example is shown

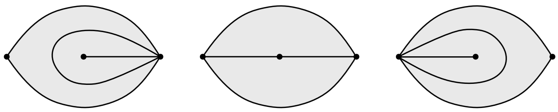


Figure 1.6. The three triangulations of a digon with one puncture.

in Figure 1.6, displaying the three different types of triangulations of a once-punctured digon.

1.3 Flips of arcs in triangulations

The number of arcs in a triangulation is an invariant of the surface. In fact, any two triangulations of (S, M) can be linked to each other by a sequence of "flips", where in each step, we replace a single arc by another arc in the surface (this is a consequence of Theorem 1.3.5 below). We first introduce notation for the arcs of a self-folded triangle.

Notation 1.3.1. Consider a self-folded triangle in a triangulation of a marked surface (S, M) as in Figure 1.7c. We call the outer arc (starting and ending at the same vertex of M) the *loop* and the enclosed arc the *radius* of the self-folded triangle. At least one endpoint of the radius is a puncture of M. If γ is the radius, we write $\ell(\gamma)$ to denote the loop of the self-folded triangle.

Definition 1.3.2. Let (S, M) be a marked surface with triangulation T, let γ be an arc of the triangulation T.

(i) If γ is not an arc of a self-folded triangle, γ defines a quadrilateral in the surface. This quadrilateral might be degenerate in the sense that it has less than four vertices. Let γ' be the other diagonal in this quadrilateral. Then the *flip* of γ is defined to be γ'. See Figures 1.7a and 1.7b.

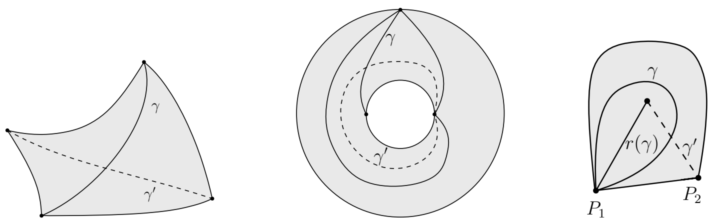
(ii) If γ is the loop of a self-folded triangle, it is enclosed in a punctured digon. Let γ' be the arc in this digon connecting the puncture to the other point of the digon. Then the flip of γ is defined to be γ'. See Figure 1.7c.

The radius of a self-folded triangle cannot be flipped. One can remedy this by introducing tagged flips, see Definition 1.5.3 below.

Definition 1.3.3. The *flip graph* of a marked surface (S, M) is the graph $E(S, M)$ whose vertices are the triangulations of the surface and whose edges are the flips.

Example 1.3.4. The flip graph of the triangle with one puncture from Example 1.2.4 is shown in Figure 1.8. Note that some of the vertices are only two-valent since we cannot flip the radius of a self-folded triangle. We will revisit this example in Section 1.5 below where we show how to complete $E(S, M)$ to a regular graph (see Figure 1.16).

A result of Hatcher, [70], implies the following:



(a) Flip **(b)** Flip in degenerate quadrilateral **(c)** Loop flip

Figure 1.7. Flip of various arcs γ. The flipped arc γ' is dashed.

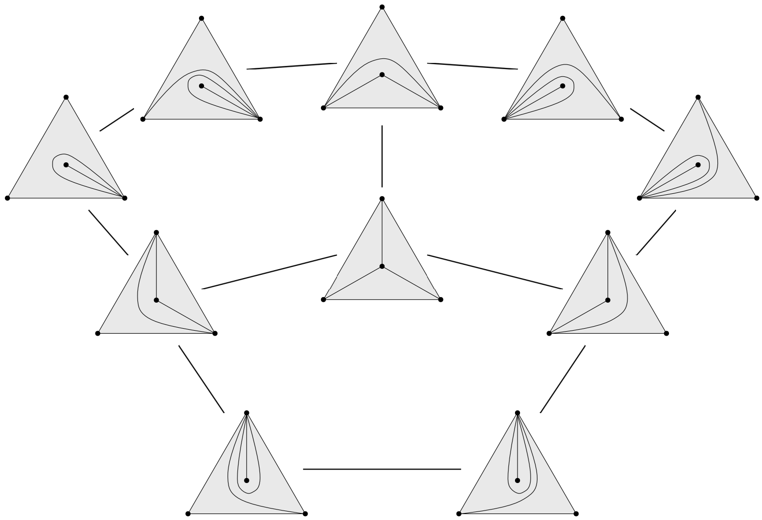


Figure 1.8. The graph $E(S, M)$ for a triangle with one puncture.

Theorem 1.3.5. *Let (S, M) be a marked surface. Then $E(S, M)$ is connected.*

Exercise 3. Work out the flip graph $E(S, M)$ for an annulus with one marked point on each boundary component, cf. Figure 1.10a.

It is sometimes convenient to assume that there are no self-folded triangles in a triangulation, so that every arc is flippable. The next result tells us that this is not a restriction.

Lemma 1.3.6. *Any surface (S, M) has a triangulation without self-folded triangles.*

As a consequence: any triangulation of (S, M) can be transformed into one without self-folded triangles by a sequence of flips.

Sketch. Suppose first that the genus of the surface is zero.

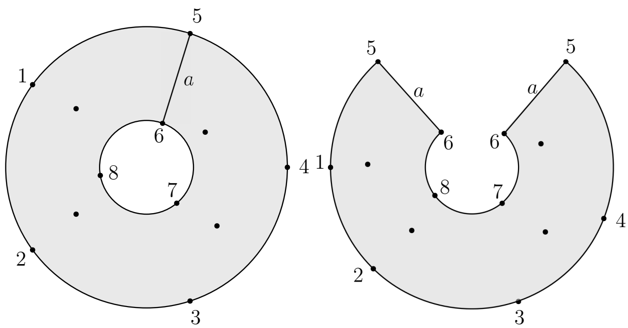


Figure 1.9. Cutting along the arc a for a surface with two boundary components.

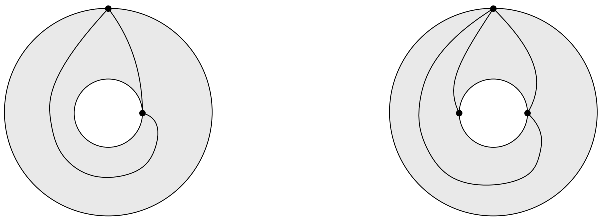
- If the surface has no boundary, we cyclically connect the punctures by non-intersecting arcs, this provides a triangulation as claimed.
- If (S, M) has b boundary components, we draw non-intersecting arcs connecting component i with component $i + 1$, for $i = 1, \ldots, b - 1$. Figure 1.9 shows an example with two components, the arc a connects the two. We cut open along these connecting arcs to produce a surface with a single boundary component. This surface is a disk with punctures that can be triangulated avoiding self-folded triangles. (One way to do this is to connect every marked point on the boundary to a puncture, creating compatible arcs.) Then, we glue the surface back together, along the connecting arcs. One observes that this does not create any self-folded triangles.

If the genus of the surface is positive, we use induction on the genus. Consider the surface S' obtained from gluing a disk to each connected component of the boundary. Note that S' has punctures: Any puncture of S is still a puncture in S'. If S has no punctures, then there exists at least one marked point P on a boundary component. By gluing a disk to this boundary component, the marked point P becomes a puncture. Since $g > 0$, there exists a closed curve γ in (S, M) around a handle in S', starting and ending at a puncture of S'. (As argued, this puncture is a marked point on a connected component of the boundary of S or a puncture of S.) Now we cut S' along γ to reduce the genus and use the induction. ■

1.4 The rank of a marked surface

Lecture 2

A formula for the number of arcs in a triangulation can be derived using the Euler characteristic of the surface, see [50, Proposition 2.10].



(a) Annulus with one point on each boundary. **(b)** Annulus with one and two points.

Figure 1.10. Two examples of triangulations in small rank (type $\tilde{A}$).

Proposition 1.4.1. *The number n of arcs in any triangulation of (S, M) is equal to*

$$n = 6g + 3b + 3p + c - 6 \tag{1.1}$$

Lemma 1.4.2. *Let (S, M) be a marked surface and let $P_0 \in M$ be a puncture, let T be a triangulation of (S, M) with n arcs. Let $M' := M \setminus \{P_0\}$. Then any triangulation of (S, M') has $n - 3$ arcs.*

Corollary 1.4.3. *For every $n > 0$, up to homeomorphism, there are finitely many marked surfaces whose triangulations consist of n arcs.*

Definition 1.4.4. Let (S, M) be a marked surface and n the number of arcs in any triangulation of it, as in (1.1). We say that n is the *rank of* (S, M).

Example 1.4.5. Using Proposition 1.4.1, one can determine all surfaces (up to homeomorphism) with rank up to three. We already discussed several of them in Example 1.2.9.

Rank 1. The surface is a quadrilateral.

Rank 2. The surface is one of the following:

- a pentagon,
- a digon with one puncture,
- an annulus with one marked point on each boundary component.

Rank 3. The surface is one of the following:

- a hexagon,
- a triangle with one puncture,
- an annulus with one, respectively, two points on the boundary components,
- a torus with one puncture.

We will revisit these surfaces in Exercise 9 and Remark 1.6.4.

Two concrete examples of marked surfaces in rank 2 and 3, respectively are shown in Figure 1.10. By Corollary 1.4.3, there are infinitely many triangulations in these

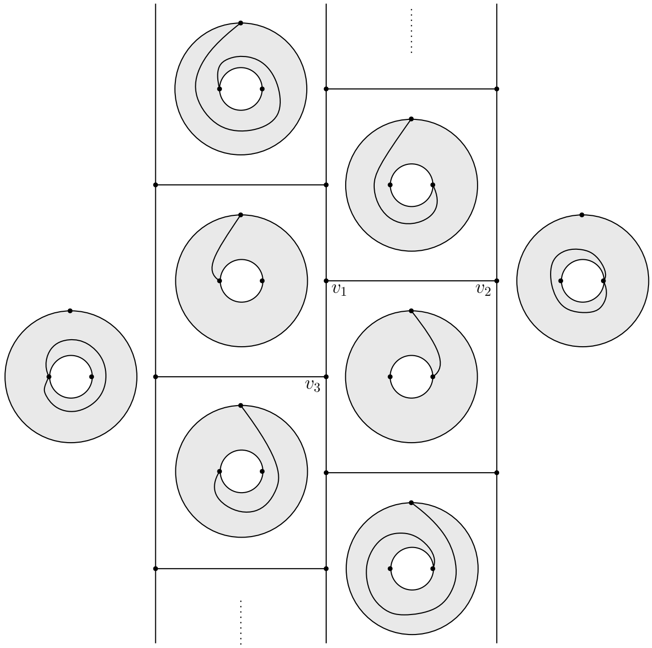


Figure 1.11. The flip graph for an annulus with one and two points on the boundaries.

cases. In the following example, we work out the exchange graph $E(S, M)$ for the surface in 1.10b.

Example 1.4.6. Let (S, M) be an annulus with one marked point on the outer boundary and two marked points on the inner boundary, as in Figure 1.10. This surface has infinitely many arcs and infinitely many triangulations. We distinguish two types of arcs: arcs with both endpoints on the same boundary and arcs connecting the two boundary components. The former are called *peripheral arcs*. This surface only has two peripheral arcs, they are drawn in the left most and right most column of the figure.

We give the exchange graph $E(S, M)$ by drawing the arcs of the surface (the faces of the flip graph), see Figure 1.11. Any two adjacent faces are pairs of compatible arcs. The vertices v_i are all of valency three and indicate three pairwise compatible arcs, i.e., the v_i are the triangulations of (S, M), the edges indicate the flip.

One can see that the left most and the right most arc in the figure are compatible with infinitely many pairs of arcs in the respective neighboured columns. Figure 1.12 shows three of the associated triangulations.

1.5 Tagged arcs

This section was mostly skipped in the class!

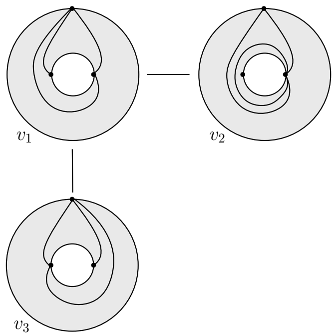


Figure 1.12. The triangulations for vertices v_1, v_2, v_3 of the flip graph from Figure 1.11.

So far, we have introduced triangulations and flips of arcs of a triangulation. We have seen that the flip graph of a marked surface is not regular in general, as the radius in a self-folded triangle cannot be flipped (see Figure 1.8 for an example). However, we would like to use triangulations as a source of seed patterns and seed patterns are defined from the regular graph, cf. Definition 1.7.13.

To fix this and ensure that every arc of a triangulation can be flipped, one can work with tagged arcs and tagged triangulations as introduced in [50, Definition 7.1].

Definition 1.5.1. Let (S, M) be a marked surface. Let γ be an arc of S as in Definition 1.2.1. Let P_1 and P_2 be the endpoints of γ, $P_i \in M$. These may coincide.

(1) A *tagging* for γ is a pair of decorations on γ near the endpoints P_1 and P_2. This decoration is either "plain" (or "unnotched") or "notched" (indicated with a red mark on the arc near the endpoint). If an endpoint of γ is on the boundary, the tagging is always plain. If an endpoint is a puncture, the tagging can be plain or notched. If both endpoints of γ coincide, the taggings of γ are the same at both ends.

(2) A *tagged arc* in S is an arc equipped with a tagging.

(3) Two tagged arcs γ and γ' are *compatible* as tagged arcs if they satisfy one of the following:

- (i) The underlying untagged arcs are compatible in the sense of Definition 1.2.3 and the arcs do not have a common endpoint at a puncture.
- (ii) The underlying untagged arcs are different, they have a common endpoint at a puncture, and they have the same tag at this puncture.
- (iii) The underlying untagged coincide, γ and γ' are different and their tags differ exactly at one endpoint.

(4) A *tagged triangulation* is a maximal collection of compatible tagged arcs.

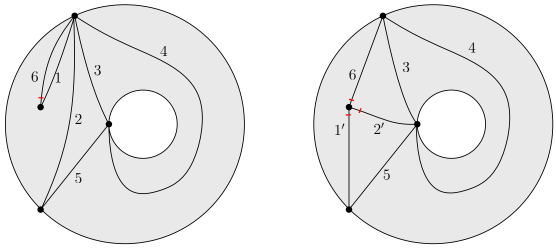


Figure 1.13. Two tagged triangulations of an annulus with one puncture.

By (3), if a tagged triangulation has arcs with different tags at a puncture, there are only two arcs ending at this puncture and their other endpoints also coincide. In all other cases, the arcs ending at a puncture have to be of the same tag. Figure 1.16 illustrates this with the tagged triangulations of a punctured triangle.

Example 1.5.2. Two tagged triangulations for the triangulations of an annulus with two marked points on the outer boundary, one on the inner boundary and one puncture are in Figure 1.13 (cf. Example 1.4 for untagged triangulations of this surface). The one on the left is the tagged triangulation of the triangulation in Figure 1.4a. Note that arc 1 of the triangulation on the left is not compatible with the arcs $2', 6'$ on the right.

Definition 1.5.3. Let T be a tagged triangulation of a marked surface (S, M). Let γ be a tagged arc of T.

(1) The *tagged flip of* γ in T is the arc γ' of (S, M) defined as follows:

(i) Let γ be a diagonal of a quadrilateral given by T. Then we replace γ by the other diagonal in this quadrilateral, say δ. If one of the endpoints of δ is a puncture P, we add a notch near P if T contains arcs with a notched near P. Otherwise, we keep it plain. We denote the result by γ'.

(ii) Let γ be a tagged arc which has a triangle on one side and a punctured digon on the other side. See Figure 1.14. This digon has two tagged arcs β_1, β_2 with the same endpoints and the two arcs have different tags at the puncture inside the digon. We label the vertices of the triangle by P_1, P_2, P_3 and the puncture by P_4 such that P_1 is a common endpoint of γ, β_1 and β_2, and that P_1 and P_3 are the marked points of the punctured digon.
The tagged flip replaces γ by an arc γ' from P_1 to P_2, which is a boundary arc of a punctured digon with vertices P_1, P_2 (and puncture P_4). The tags of γ' are determined by the tags of existing arcs of T at these two endpoints.

(iii) Let γ be one of a pair of tagged arcs (β_1, β_2) whose underlying untagged versions are the same. We use the notation of Figure 1.14. Then the two

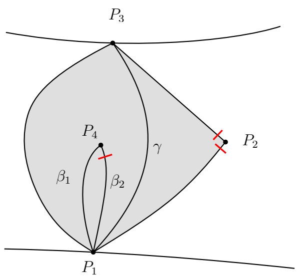
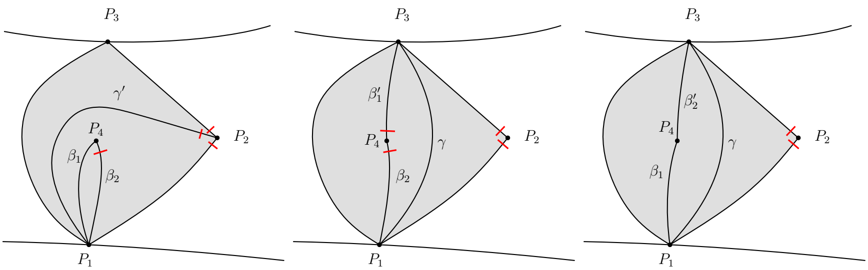


Figure 1.14. Part of a tagged triangulation near a pair of arcs with different tags at P_4.



(a) After flipping γ. **(b)** After flipping β_1. **(c)** After flipping β_2.

Figure 1.15. The effect of the tagged flip on arcs γ, β_1, β_2.

arcs are inside a punctured digon formed by T and have different tags at one endpoint, either at P_1 or at P_4, and the same tag at the other end by Definition 1.5.1 (3). In the figure, the different tagging is at P_4, the other case works analogously. Let P_3 be the other marked point of the digon enclosing (β_1, β_2).

Flipping γ replaces it by an arc γ' with endpoints P_1 and P_2 inside the punctured digon (Figures 1.15b, 1.15c). The tags of γ' are determined by the tags of existing arcs of T at the two endpoints of the new arc.

(2) The *tagged arcs graph* $E^{\bowtie}(S, M)$ is the graph whose vertices are the tagged arcs and whose edges are the tagged flips.

Exercise 4. Work out a tagged flip sequence linking the two tagged triangulations of Figure 1.13.

Exercise 5. Check that every arc of a tagged triangulation can be flipped.

Example 1.5.4. Figure 1.16 shows graph $E^{\bowtie}(S, M)$ for the punctured triangle. Note that this completes the graph of Figure 1.8 into a regular graph.

Exercise 6. Let (S_1, M_1) be a once-punctured triangle and (S_2, M_2) a hexagon.

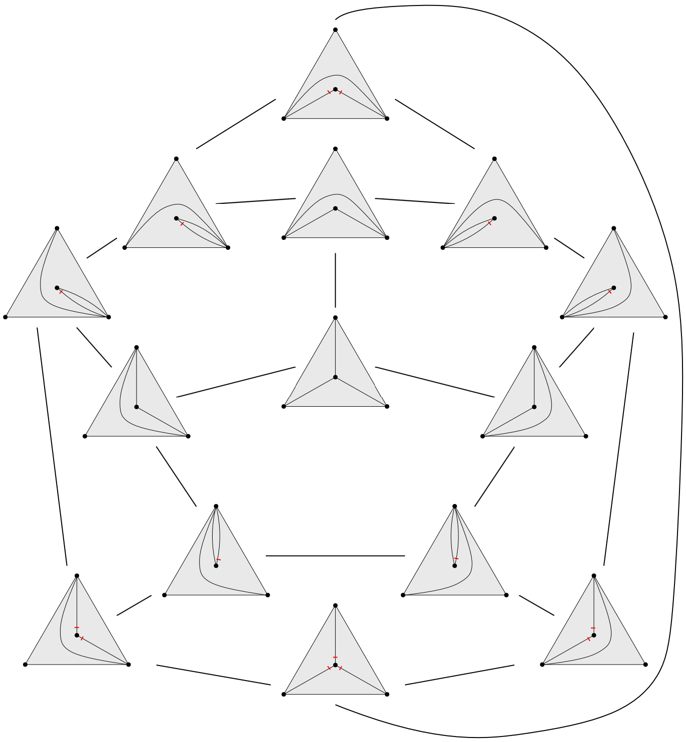


Figure 1.16. The graph $E^{\bowtie}(S, M)$ for the once-punctured triangle.

(1) Compare the graph $E^{\bowtie}(S_1, M_1)$, $E(S_2, M_2)$ and the (untagged) graph of the punctured triangle from Figure 1.8.

(2) What are the faces of the three graphs?

In the sequel, we work with the untagged versions of arcs and triangulations and allow self-folded triangles, unless specified otherwise.

1.6 The quiver of a triangulation

To define cluster algebras from surface triangulations, we use the adjacency in the triangulation, i.e., the relative position of the arcs in it, and associate a quiver to the relative positions of the arcs.

We first recall the definition of a quiver - this is an oriented graph:

Definition 1.6.1. A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 are the *vertices* of Q, Q_1 the set of *arrows* of Q and, where $s, t : Q_1 \to Q_0$ are maps which associate to each arrow $\alpha \in Q_1$ its *source* $s(\alpha) \in Q_0$ and its *target* $t(\alpha) \in Q_0$. We often simply write Q or (Q_0, Q_1) for a quiver. We usually denote an arrow α with source $s(\alpha) = a$

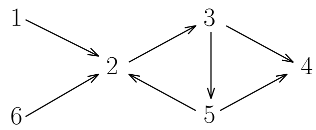


Figure 1.17. The quiver Q_T for the triangulation in Figure 1.4a.

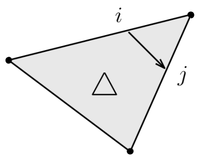
and target $b = t(\alpha)$ by

$$\alpha : a \longrightarrow b \quad \text{or by} \quad a \xrightarrow{\alpha} b.$$

Unless specified otherwise, the quivers we consider are finite, i.e., they have finitely many vertices and finitely many arrows. We use quivers to describe "seeds" (as in Definition 1.6.5 (2)). The idea is that the vertices of a quiver correspond to the cluster variables, while the oriented edges encode the mutation rule.

Definition 1.6.2. Let T be a triangulation of a marked surface (S, M) with n arcs, labelled $1, 2, \dots, n$. We define the *quiver* $Q = Q_T$ *of the triangulation* as follows: The vertices of Q_T are the arcs of T. We draw an arrow

(a) $i \longrightarrow j$ if j follows i clockwise inside a common triangle which is not self-folded. (In other words, the arcs i and j belong to the same triangle, they have a common endpoint and j is clockwise from i).



(b) $i \longrightarrow j$ if j is a radius enclosed by the loop $\ell(j)$ and $\ell(j)$ follows i clockwise inside a common triangle, with $i \neq j, i \neq \ell(j)$.

(c) $j \longrightarrow i$ if j is a radius enclosed by the loop $\ell(j)$ and i follows $\ell(j)$ clockwise inside a common triangle, with $i \neq j$, $i \neq \ell(j)$. (An Example is the arrow $1 \longrightarrow 2$ in Figure 1.17, for the triangulation in Figure 1.4a).

At the end, we remove a maximal collection of 2-cycles.

Note that the quiver Q_T has no loops and no 2-cycles.

Exercise 7. Work out the quiver of the triangulation of the punctured torus from Example 1.2.5 (see Figure 1.3).

Exercise 8. Draw the quiver of the triangulation v_2 in Figure 1.12.

With the definition of a quiver of a triangulation, we can introduce the notion of the type of a surface:

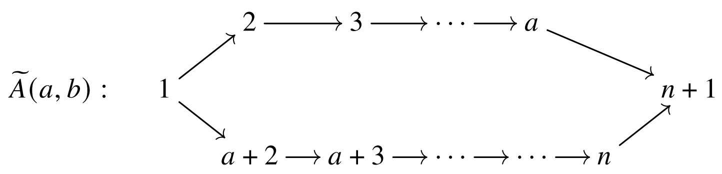


Figure 1.18. The quiver $\widetilde{A}(a,b)$ of extended type $\widetilde{A}_n$ with $n = a + b$.

Definition 1.6.3. Let (S, M) be a marked surface. Let G, G_1, G_2 be simply-laced Dynkin diagrams or simply-laced extended Dynkin diagrams. We say that *the surface S is of type G, respectively of type $G_1 \times G_2$*, if there exists a triangulation T of it such Q_T is an orientation of G, respectively of $G_1 \times G_2$.

We also need the following notation: If a quiver arises from orienting an extended diagram $\widetilde{A}_n$ such that there is exactly one source and one sink, with a arrows in one direction and b arrows in the other direction, we say that the graph is of type $\widetilde{A}(a,b)$, see Figure 1.18.

Exercise 9. In Example 1.4.5 the surfaces of rank up to three were listed. Prove that the type of these surfaces is as claimed in the following list:

Rank 1. The quadrilateral is of Dynkin type A_1.

Rank 2.

- The pentagon is of Dynkin type A_2,
- a digon with one puncture is of Dynkin type $A_1 \times A_1$,
- an annulus with one marked point on each boundary component is of type $\tilde{A}_1$. More precisely, the quiver of any triangulation of this annulus has two vertices $1, 2$ and two arrows from 1 to 2 (this is called $\widetilde{A}(1,1)$, see Figure 1.18).

Rank 3.

- The hexagon is of Dynkin type A_3,
- a triangle with one puncture is of Dynkin type A_3,
- an annulus with one and two points on the boundary components, is type $\tilde{A}_2$. More precisely, such an annulus has a triangulation T such that the quiver Q_T is $\widetilde{A}(1,2)$, see Figure 1.18.

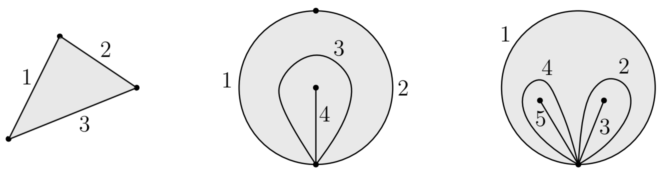
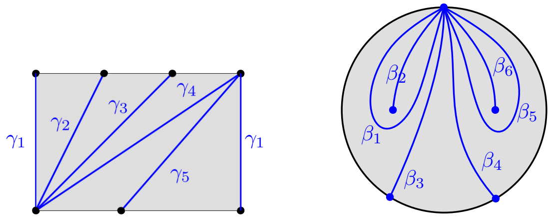


Figure 1.19. Three types of puzzle pieces

Exercise 10. Determine the quivers of the following two triangulations:



Remark 1.6.4. To summarise, we list the surfaces of rank up to three in the table below. As before, g denotes the genus, b the number of connected components on the boundary, p the number of punctures, c the number c of marked points on the boundary and n the rank. We write $\mathrm{Ann}(m_1, m_2)$ to denote an annulus with m_1 marked points on one boundary and m_2 marked points on the other boundary. Examples for the annuli in the table are in Figure 1.10.

(S, M)	g	b	p	c	n	type
$(n+3)$-gon	0	1	0	$n+3$	n	A_n
n-gon, one puncture	0	1	1	n	n	D_n
annulus $\mathrm{Ann}(m_1, m_2)$	0	2	0	$n_1 + n_2$	$n_1 + n_2$	$\widetilde{A}(m_1, m_2)$
$(n-3)$-gon with two punctures	0	1	2	$n-3$	n	$\widetilde{D}_{n-1}$
torus with one puncture	1	0	1	0	3	-

We will also use an extended version of the quiver Q_T.

Definition 1.6.5. Let T be a triangulation of (S, M), let $n = |T|$ be the rank of the surface and Q_T be the quiver of T.
(1) We add n vertices to Q_T, denoted $\boxed{n+i}$, with arrows $\boxed{n+i} \to i$, $i = 1, \ldots, n$. We denote this extended quiver by $\widetilde{Q}_T$.
(2) Let $\underline{x}_T := (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n})$, with variables x_i. We call $(\underline{x}_T, \widetilde{Q}_T)$ the *(extended) seed of* T.

1.7 The cluster algebra of a quiver

In this section, we always assume that quivers have no loops (no arrows starting and ending at the same vertex) and no 2-cycles (no pair of arrows $i \to j$ and $j \to i$). We also often assume that there are no arrows between frozen vertices.

Definition 1.7.1. Let $Q = (Q_0, Q_1)$ be a quiver with vertices $Q_0 = \{1, 2, \ldots, n, \ldots, m\}$, with $m \geq n > 0$. We say that Q is a *cluster quiver* if the following are satisfied:

(i) The quiver Q has no loops;

(ii) There are no 2-cycles in Q;

(iii) There are no arrows between i and j if $i, j > n$.

The vertices $\{1, 2, \ldots, n\}$ of Q are called *mutable* and the vertices $\{n+1, \ldots, m\}$ are called *frozen*.

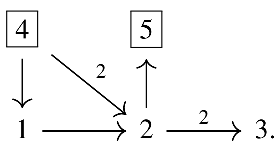
Definition 1.7.2. Let $m \geq n > 0$ and Q be an extended quiver with n mutable and $m - n$ frozen vertices. Let $x_1, \ldots, x_m$ be variables. We call $\underline{x} := (x_1, \ldots, x_m)$ an (extended) *cluster*, with *cluster variables* $x_1, \ldots, x_n$ and *frozen variables* $x_{n+1}, \ldots, x_m$. We call the tuple $(\underline{x}, Q)$ an (extended) seed.

Cluster quivers are not required to be connected. However, they will mostly be so.

Remark 1.7.3. The arrows between vertices do not play a role in the mutation process, and it is sometimes convenient to allow arrows and 2-cycles between frozen arrows. In particular, "dimer quivers" are examples of cluster quivers with 2-cycles between frozen arrows, and they may be used to define cluster algebras. We can reduce such quivers by successively removing arrows in 2-cycles on boundary vertices, keeping only one arrow between the two vertices involved. This approach is convenient when working with cluster categories. However, unless specified otherwise, we stick to condition (iii) in Definition 1.7.1.

We usually draw quivers as graphs in the plane, sometimes with crossing arrows. Frozen vertices are often indicated as boxed vertices, see Example 1.7.4. When drawing a quiver with multiple arrows between two given vertices, we often only draw a single arrow and indicate the number of arrows as a label on the arrow.

Example 1.7.4. The following is a cluster quiver with $n = 3, m = 5$. The frozen vertices are the vertices 4, 5, drawn in boxes.



The two arrows from vertex 2 to vertex 3 and the two arrows from vertex 4 to vertex 2 are indicated by the corresponding labels on the arrows.

Remark 1.7.5. We can also associate cluster algebras to matrices with integer entries, taking (part of) the adjacency matrix of the quiver. More precisely, the matrices are $m \times n$-matrices. The top $n \times n$ submatrix is skew-symmetric and the adjacency of mutable vertices with frozen vertices is encoded in $m - n$ rows with n entries (since there are no arrows between them, this is sufficient). We call this an extended skew-symmetric matrix.

For the quiver from Example 1.7.4, the matrix is

$$B(Q) = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Using matrices allows to define cluster algebras more generally: they can be defined for "skew-symmetrizable" extended integral matrices.

Exercise 11. Check that the matrix of any cluster quiver is an extended skew-symmetric matrix.

The definition of mutation uses the notion of a 2-*path* through a vertex k of a cluster quiver Q. A 2-path is a path $i \to k \to j$ obtained from composing two incident arrows $i \to k$ and $k \to j$ of Q.

Definition 1.7.6. Let Q be a cluster quiver with mutable vertices $\{1, 2, \ldots, n\}$ and frozen vertices $\{n+1, \ldots, m\}$. Let $k \leq n$. The *quiver mutation* $\mu_k(Q)$ of Q in *direction* k, is defined as follows:

(1) For every 2-path $i \to k \to j$, we add an arrow $i \to j$ whenever i and j are not both frozen.

(2) We reverse all arrows at k,

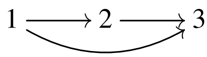
(3) We delete a maximal collection of 2-cycles (deleting pairs of arrows).

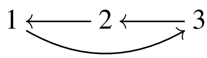
The first step introduces a "shortcut" in the quiver for every 2-path passing through k (modulo frozen vertices: we do not introduce shortcuts between frozen vertices). Adding the shortcuts may create 2-cycles. In order for the result under quiver mutation to be another cluster quiver, we have to remove the 2-cycles in the third step. We note that mutation does not create any loops since Q does not contain any 2-cycles.

Definition 1.7.7. We say that two cluster quivers Q, Q' are *mutation equivalent* if Q' can be obtained from Q through a sequence of mutations. The *mutation equivalence class* of a cluster quiver Q is the set of cluster quivers obtained through arbitrary sequences mutations from Q.

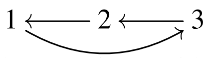
Definition 1.7.8. Let Q be a cluster quiver. The graph whose vertices are the quivers mutation equivalent to Q and whose edges are given by single mutations is the *exchange graph of* Q. We denote it by $E(Q)$.

Example 1.7.9. Consider the quiver $Q : 1 \to 2 \to 3$ with three mutable vertices, i.e., $m = n = 3$. We work out $\mu_2(Q)$:

step (1) 

step (2) 

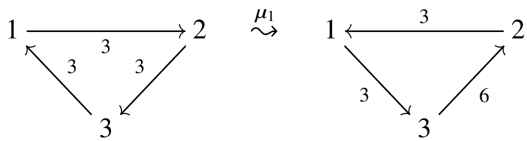
This example does not involve step (3) since after step 2, the quiver does not contain a 2-cycles. So we have

$\mu_2(Q) =$  .

Exercise 12. Here we consider two cluster quivers with $m = n = 3$.

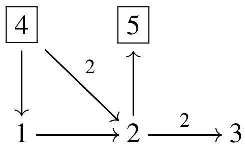
(1) Consider the quiver $\mu_2(Q)$ from Example 1.7.9 and mutate it at every vertex, i.e., work out $\mu_i\mu_2(Q)$ for $i = 1, 2, 3$.

(2) **Lecture 3**
Applying mutation in direction 1 to the quiver Q on the left, we obtain the quiver on the right:

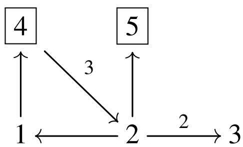


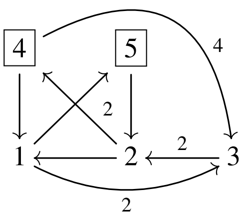
Work out $\mu_j(Q)$ for $j = 2, 3$ and $\mu_i\mu_1(Q)$ for $i = 2, 3$. Can you find a pattern? How does the quiver develop when you do more mutations?

Example 1.7.10. Let $m = 5, n = 3$ and consider the quiver Q with frozen vertices 4 and 5:



We work out $\mu_i(Q)$ for $i = 1, 2$.

$\mu_1(Q) =$ 

$\mu_2(Q) =$ 

We know how to mutate quivers. Now we explain how to mutate extended seeds, i.e., we have to say how to mutate the variables $x_1, \ldots, x_m$. When we mutate them, we work in the field of fractions $\mathbb{Q}(x_1, \ldots, x_m)$.

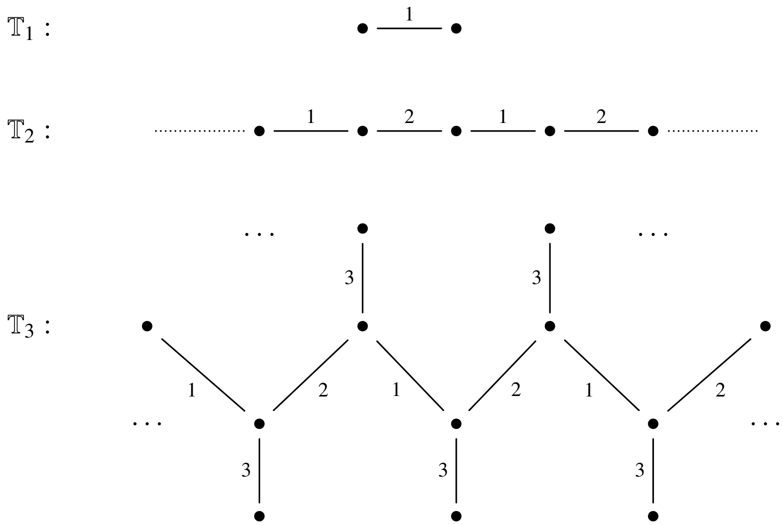


Figure 1.20. The trees $\mathbb{T}_n$ for $n = 1, 2, 3$.

Definition 1.7.11. Let Q be a cluster quiver and let $\underline{x} = (x_1, \ldots, x_m)$ be an extended cluster with n mutable vertices and $m - n$ frozen vertices, $m \geq n$. For $1 \leq k \leq n$, the (seed) *mutation in direction* k of the extended seed $(\underline{x}, Q)$ is defined as

$$\mu_k(\underline{x}, Q) = (\underline{x}', Q')$$

where $Q' = \mu_k(Q)$ and where the extended cluster $\underline{x}'$ only differs from $\underline{x}$ in position k, with x'_k being defined via the exchange relation

$$x_k \cdot x'_k := \prod_{i \to k} x_i + \prod_{i \leftarrow k} x_i.$$

So $\underline{x}' = (x_1, \ldots, x'_k, \ldots, x_n, \ldots, x_m)$ with $x'_k = \frac{\prod_{i \to k} x_i + \prod_{i \leftarrow k} x_i}{x_k}$.

We associate a "seed pattern" to $(\underline{x}, Q)$ and use it to define a cluster algebra.

Notation 1.7.12. We denote by $\mathbb{T}_n$ an n-regular tree with edges labelled by $1, \ldots, n$ such that at every vertex, the edges incident with it have all n labels. Figure 1.20 illustrates $\mathbb{T}_n$ for small n.

In general, $\mathbb{T}_n$ is infinite, if $n > 1$. We use the tree $\mathbb{T}_n$ to keep track of different mutations in cluster algebras.

Definition 1.7.13. A *seed pattern for* $\mathbb{T}_n$ is an assignment of a labelled seed $(\underline{x}(t), Q(t))$ to every vertex $t \in \mathbb{T}_n$ such that there is an edge labelled by k between vertices t and t' of $\mathbb{T}_n$ if and only if $\mu_k((\underline{x}(t), Q(t))) = (\underline{x}(t'), Q(t'))$.

A seed pattern is uniquely defined by any one of its seeds. We also call it the *seed pattern of a (given) seed.*

Let $\mathcal{X}$ be the union of all cluster variables and frozen variables of the seed pattern of the seed $(\underline{x}, Q)$.

Notation 1.7.14. Let $(\underline{x}(t), Q(t))_{t\in\mathbb{T}_n}$ be a seed pattern for $0 < n \le m$ and denote by

$$\mathcal{X} := \bigcup_{t\in\mathbb{T}_n} \underline{x}(t)$$

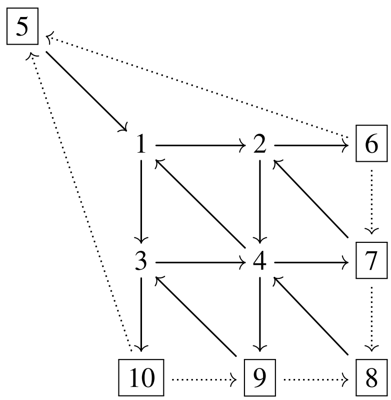
the set of all cluster variables appearing in all seeds of the pattern, together with the frozen variables.

Definition 1.7.15. Let Q be a cluster quiver with n mutable vertices and $m - n \ge 0$ frozen vertices. The *cluster algebra* $\mathcal{A}(Q)$ *of* Q is defined to be the cluster algebra (of rank n) of the seed pattern defined by $(\underline{x}, Q)$.

Definition 1.7.16. Let Q be a cluster quiver with n mutable and $m - n$ frozen vertices, $m \ge n \ge 1$. The *cluster algebra* $\mathcal{A} = \mathcal{A}(Q)$ over $R := \mathbb{Q}[x_{n+1}, \dots, x_m]$ associated with the seed pattern for $\mathbb{T}_n$ is the R-subalgebra $\mathcal{A} = R[\mathcal{X}]$ of the field of fractions $\mathbb{F}$ generated by all cluster variables and the frozen variables. The number n of mutable vertices is called the *rank of the cluster algebra.*

While it is sometimes convenient to invert the frozen variables $x_{n+1}, \dots, x_m$, we will not do this unless mentioned otherwise.

Example 1.7.17. Let $n = 4$ and $m = 10$ and consider the seed given by the quiver:



The frozen variables are indicated by the boxed numbers. Here, we allow arrows between frozen arrows, indicating them by dotted lines. This is a seed for the Grassmannian cluster algebra structure of Gr(3, 6). In terms of Plücker coordinates, the extended seed $\underline{x} = (x_1, \ldots, x_{10})$ is

$$\underline{x} = (p_{124}, p_{134}, p_{125}, p_{145}, p_{123}, p_{234}, p_{345}, p_{456}, p_{156}, p_{126}).$$

Scott proved [110] that $\mathcal{A}(Q) \cong \mathbb{C}[\mathrm{Gr}(3,6)]$ (in fact, she gave a way to construct a seed for any $\mathrm{Gr}(k,n)$).

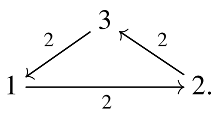
It is known that a seed of the Grassmannian cluster algebra structure of $\mathrm{Gr}(k,n)$ has $(k-1)(n-k-1)$ mutable and n frozen vertices, so here, $10 = (k-1)(n-k-1)+n = 2 \cdot 2 + 6$ vertices of Q. One can show that there are only finitely many clusters mutation equivalent to it, namely 50. This seed and all seeds obtained through arbitrary sequences of mutations from it are described in [69, Subsection 3.1.2]: There are 50 clusters in this case.

One can use Bernhard Keller's quiver mutation applet[1] to get seeds for the Grassmannian cluster algebra $\mathcal{A}(k,n)$. In the downloaded version of the applet, there are two types of seeds for given (k,n), one built up from oriented triangles and one containing oriented quadrilaterals, using the menu item "Activate" under "Clusters".

In contrast, the following cluster algebra of rank 3 has infinitely many cluster variables This example illustrates a link between cluster algebras and number theory. It appears in [63, Section 2.1] and also in [51, Chapter 3.4] where further examples illustrating the link to number theory are shown.

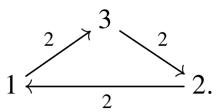
Example 1.7.18. Consider the so-called Markov quiver on three vertices: it is a quiver with $n = m = 3$ and with two arrows $i \longrightarrow i+1$ for $i = 1, 2, 3$.

Q :



Mutation of Q in direction 1 yields the quiver

$\mu_1(Q)$:



In fact, for any i, the mutated quiver $\mu_i(Q)$ is isomorphic to Q. So up to relabelling the vertices, the Markov quiver is invariant under mutation. The exchange relations are:

$$\begin{aligned} x_1 \cdot x_1' &= x_2^2 + x_3^2, \\ x_2 \cdot x_2' &= x_1^2 + x_3^2, \\ x_3 \cdot x_3' &= x_1^2 + x_2^2. \end{aligned}$$

[1] https://webusers.imj-prg.fr/~bernhard.keller/quivermutation/

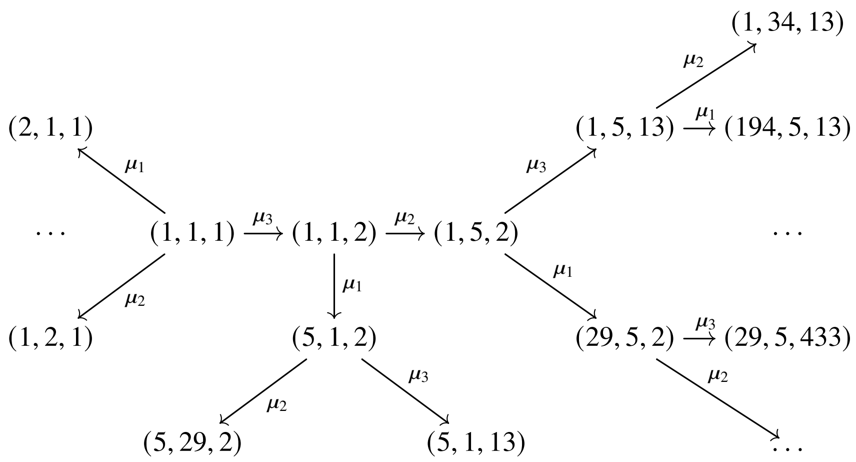


Figure 1.21. Triples of integers arising from the Markov quiver.

To see that even though the quiver is invariant under mutation, there are infinitely many variables, we specialize the initial cluster variables to 1, i.e., we set $x_1 = x_2 = x_3 = 1$ and study the effect of mutation on these values. Since all cluster variables can be written as Laurent polynomials (Theorem 1.7.21) with positive coefficients (Theorem 1.7.22) in the initial variables and since we specialized the initial variables to 1, all cluster variables specialize to positive integers. The graph in Figure 1.21, shows the values obtained from the initial cluster for a few mutation sequences.

One can prove that for every cluster (x_1, x_2, x_3), the equation

$$x_1^2 + x_2^2 + x_3^2 = 3x_1x_2x_2$$

holds. As a consequence, if we specialize the initial cluster to 1, any cluster variable is a positive integer and for any cluster (x_1, x_2, x_3), the resulting tuple $(a_1, a_2, a_3) \in \mathbb{Z}^3$ satisfies the Markov equation

$$a_1^2 + a_2^2 + a_3^2 = 3a_1a_2a_3.$$

Mutation transforms Markov triples into Markov triples and arbitrary sequences of mutations give an infinite set of triples in $\mathbb{Z}^3$ which all satisfy the Markov equation, cf. [51, Example 3.4.1]. In particular, there are infinitely many cluster variables for this quiver.

The next example shows a link to frieze patterns in the sense of Conway and Coxeter, [36, 37].

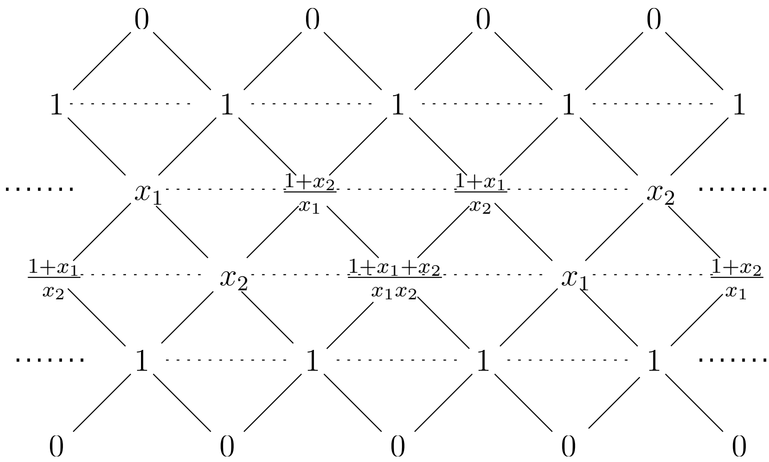


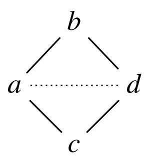
Figure 1.22. The frieze formed by the variables of the cluster algebra $\mathcal{A}(1,1)$.

Example 1.7.19. Let $n = m = 2$ and consider the cluster $\{(x_1, x_2), 1 \to 2\}$ of the cluster algebra $\mathcal{A}(1,1)$ (Example 0.1.1). We consider the mutation sequence $\mu_1\mu_2\mu_1$:

$$(x_1,x_2) \xrightarrow{\mu_1} (\tfrac{1+x_2}{x_1}, x_2) \xrightarrow{\mu_2} (\tfrac{1+x_2}{x_1}, \tfrac{1+x_1+x_2}{x_1x_2}) \xrightarrow{\mu_1} (\tfrac{1+x_1}{x_2}, \tfrac{1+x_1+x_2}{x_1x_2})$$

We use the clusters as diagonals in a grid, extending with 1's and 0's at both ends, obtaining a grid of six rows which are shifted with respect to each other: We start with a row of 0's followed by a row of 1s. Then we write the variables of the cluster (x_1, x_2) in the second and in the third row, respectively, on a SE diagonal of the pattern and end with a row of 1's and a row of 0's. See Figure 1.22. When going from x_1 to the right, along the dotted line, we apply μ_1 and replace x_1 by the cluster variable $\mu_1(x_1)$. We now have the cluster $(\frac{1+x_2}{x_1}, x_2)$ on an SW diagonal in the pattern. When going from x_2 to the left, along the dotted line, we mutate x_2 in the first cluster, i.e., we put the cluster variable $\mu_2(x_2)$ to the left of x_2. This gives us the cluster $(x_1, \frac{1+x_1}{x_2})$ on the left most SW diagonal in the figure. We repeat this in both directions. We have already seen this cluster algebra in Example 0.1.1. Apart from the two initial cluster variables, there are three more cluster variables, namely $\frac{1+x_1}{x_2}$, $\frac{1+x_1+x_2}{x_1x_2}$ and $\frac{1+x_2}{x_1}$. So the pattern we obtained through this procedure repeats horizontally after 5 steps to the right (or to the left).

A *diamond* in this pattern is formed by four neighbours as follows (potentially involving the rows of 1's and of 0s):



One checks (Exercise 13) that the relation $ad - bc = 1$ is satisfied in every diamond of the pattern.

This is an example of a *frieze pattern* - a pattern formed by rows which are infinite in both directions, which satisfies the diamond rule and which has a translational symmetry. In the above example, we also have a glide symmetry.

Because of the diamond relation, such patterns are also called SL_2-friezes.

Exercise 13. Check that in the pattern from Example 1.7.19, the relation $ad - bc = 1$ is satisfied for every diamond.

1.7.1 Main theorems

We state some of the main theorems in our setting - they are valid in more general set-ups.

Definition 1.7.20. Let Q be a cluster quiver. The cluster algebra $\mathcal{A} = \mathcal{A}(Q)$ (or the quiver Q) is said to be of

(1) *finite type*, if $\mathcal{X}$ is finite;

(2) *finite mutation type*, if the mutation equivalence class of Q is finite;

(3) *acyclic type*, if Q is mutation equivalent to a quiver without oriented cycles.

Theorem 1.7.21. [52, Theorem 3.1] *Let $\mathcal{A} = \mathcal{A}(\underline{y}, Q)$ be the cluster algebra of the extended seed $(\underline{y}, Q)$, where the variables $y_1, \dots, y_n$ of $\underline{y} = (y_1, \dots, y_m)$ are mutable $(m \geq n > 0)$ and where Q is a cluster quiver.*

Then any cluster variable x of $\mathcal{A}$ is a Laurent polynomial with integer coefficients in the y_i. More precisely, there are $f \in \mathbb{Z}[y_1, \dots, y_m]$ and $r_1, \dots, r_n \geq 0$ such that

$$x = \frac{f(y_1, \dots, y_m)}{y_1^{r_1} y_2^{r_2} \cdots y_n^{r_n}}.$$

The fact that we every cluster variable x is a Laurent polynomial in the initial cluster is surprising: A priori, we deal with iterated fractions and so cluster variables are rational functions in the initial variables. However, it turns out the denominators are always monomials in the initial cluster variables. To prove this, one can use induction on the length d of a mutation sequence leading to x, cf. [51, Subsection 3.3].

Exercise 14. Let $\mathcal{A}(\underline{y}, Q)$ be as in Theorem 1.7.21. Let y be a cluster variable obtained by up to two mutations from the $y_1, \dots, y_n$. Show that y is a Laurent polynomial in the variables $y_1, \dots, y_m$ with denominator a monomial in $y_1, \dots, y_n$.

Theorem 1.7.22. *The coefficients of the Laurent polynomial in Theorem 1.7.21 are positive integers.*

This property has first been conjectured by Fomin and Zelevinsky in [52, Section 3]. The binomial exchange relations in the mutation formula only involve positive terms, however, one has to check that the positivity is preserved when reducing the rational functions appearing after mutations. It took over a decade to prove this conjecture: Lee and Schiffler prove it for all skew-symmetric cluster algebras in [88] (not necessarily of geometric type). An application of the results of [66] proves it for cluster algebras of geometric type. Intermediate results include positivity for acyclic cluster algebras ([82]) and for cluster algebras from surfaces ([96]). For more details, we refer to the introduction of the above article by Lee and Schiffler.

For the next theorem, we introduce the reduced quiver Q_{red} of Q: it is the full subquiver determined by the mutable vertices of Q, i.e., the oriented graph whose vertices are the mutable vertices of Q and whose oriented edges are the arrows of Q which start and end at mutable vertices.

Theorem 1.7.23 ([53, Theorem 1.4]). *Let $\mathcal{A} = \mathcal{A}(\underline{x}, Q)$ be a cluster algebra, with connected quiver Q. Then $\mathcal{A}$ is of finite type if and only if Q_{red} is mutation equivalent to an orientation of a Dynkin diagram of type A, D or E.*

1.8 The cluster algebra of a triangulation of a marked surface

Around 2005, Fomin, Shapiro and Thurston defined cluster algebras from marked surfaces, see [50]. They used principal coefficients but we will work with frozen variables instead. The boundary segments of the surface do not play a role for the surface cluster algebras. In the construction of cluster categories, we sometimes take boundary segments into account and they will give frozen vertices (different kinds of frozen vertices).

Let (S, M) be a marked surface of rank n, let T be a triangulation of it. We denote the arcs in T by $1, \ldots, n$. Let Q_T be the quiver of the triangulation (Definition 1.6.2). We declare every vertex of Q_T to be mutable. Recall Definition 1.6.5: we add n frozen vertices $\boxed{n+i}$ with arrows $\boxed{n+i} \to i$ to Q_T and write $\widetilde{Q}_T$ for the resulting quiver (note that the extended quiver still has no 2-cycles and no loops) and we associate the extended seed $\mathbf{x}_T := (x_1, \ldots, x_n, \ldots, x_{2n})$ to it (with first n variables mutable, the others frozen).

Definition 1.8.1. We define the *cluster algebra associated to* T and (S, M) to be

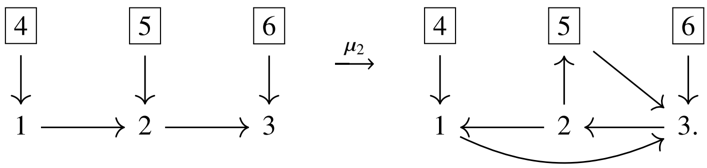
$$\mathcal{A}_T = \mathcal{A}(\mathbf{x}_T, Q_T).$$

We have $\mathcal{A}((x_1, \ldots, x_{2n}), Q) \subset \mathbb{Z}[x_{n+1}, \ldots, x_{2n}](x_1, \ldots x_n)$. We recall (Definition **??**) that mutation at k (with $1 \le k \le n$) sends the cluster $\mathbf{x} = (x_1, \ldots, x_n, \ldots, x_{2n})$ to $\mathbf{x}' = (x_1, \ldots, x_k', \ldots, x_n, \ldots, x_{2n})$ where

$$x_k \cdot x_k' := \prod_{i \to k} x_i + \prod_{i \leftarrow k} x_i.$$

Example 1.8.2. Let T be the fan triangulation of a hexagon and $Q_T = 1 \to 2 \to 3$ its quiver. We consider the mutation μ_2 of the seed $(\mathbf{x}_T, Q_T)$:

We start with the quiver on the left, with the cluster $(x_1, x_2, \ldots, x_6)$, where x_4, x_5, x_6 are coefficients/frozen:



The resulting cluster is

$$\mu_2((x_1, \ldots, x_6)) = (x_1, \frac{x_1x_5 + x_3}{x_2}, x_3, x_4, x_5, x_6).$$

The cluster algebra $\mathcal{A}_T$ is defined from a chosen triangulation of (S, M). However, as the following result shows, it is independent of that choice as mutation corresponds to flipping of arcs.

Proposition 1.8.3 ([50, Proposition 4.8 and Proposition 4.10]). *Consider a triangulation T of a marked surface (S, M). Let k be an arc in T and assume that k is not a radius of a self-folded triangle. Then, the following hold.*

(1) *If T' is obtained from T by flipping the arc k, then $Q_{T'} = \mu_k(Q_T)$,*

(2) *The mutation equivalence class of Q_T depends only on (S, M) and not on the chosen triangulation.*

We may therefore write $\mathcal{A}_{(S,M)}$ instead of $\mathcal{A}_T$.

Proof. To prove part (1), one uses the fact that the triangulation T and the surface (S, M) are built from copies of the three puzzle pieces from Figure 1.19 and studies these pieces, i.e., one proves the claim locally.

Part (2) follows from (1) and from the fact the flip graph is connected. ■

Surface cluster algebras are of infinite type in general, in fact, unless (S, M) is a disk with at most one puncture, there are infinitely many triangulations (Corollary 1.2.7). However, they are examples of cluster algebras of finite mutation type (Definition **??**) by work of Felikson, Shapiro and Tumarkin. We have already seen an example for this, namely the once punctured torus: any triangulation of it gives rise to the quiver with three vertices and two arrows from $i \to i + 1$, $i = 1, 2, 3$ (Exercise 7). Another example we have encountered are triangulations of annuli, cf. Figure 1.12:

Exercise 15. Determine the quivers Q_T which arise from triangulations of an annulus $\mathrm{Ann}(2, 1)$ with two points on one boundary and one point on the other boundary.

For the statement of the classification of finite mutation type (and for Exercise 9 below) we extend the notion of the type of a quiver:

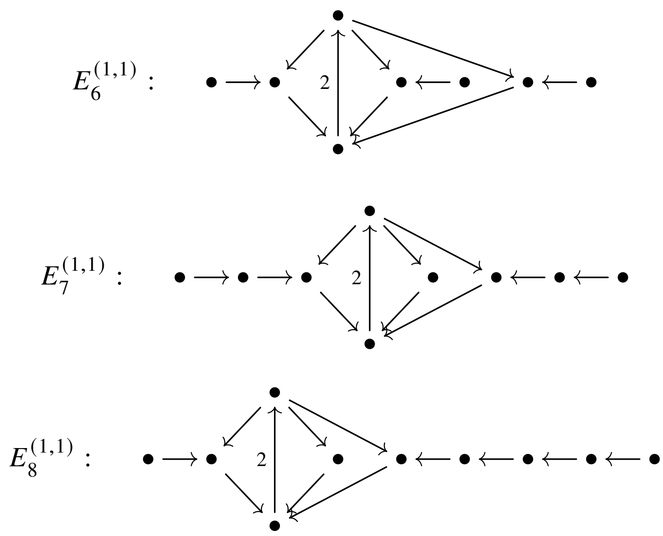


Figure 1.23. Orientations of doubly-extended type E diagrams.

Notation 1.8.4. Let G be a simply-laced Dynkin or a simply-laced extended Dynkin diagram. We say that Q is *of type* G, if Q_{red} is mutation equivalent to an orientation of G.

Quivers with only one vertex and quivers with two vertices and only one arrow are of finite type. Quivers from surface triangulations where the surface is a disk or a once-punctured disk are also of finite type (see Exercise 1).

Theorem 1.8.5 ([43, Theorem 5.6]). *Let Q be a connected cluster quiver. Assume that Q is of finite mutation type but not of finite type. Then Q is mutation equivalent to one of the following:*

(i) *A quiver with two vertices and at least two arrows.*

(ii) *A quiver of surface type where the surface is not a disk or a punctured disk.*

(iii) *A quiver of type $\widetilde{E}_n$, for $n = 6, 7, 8$.*

(iv) *A quiver Q of type $E_n^{(1,1)}$, for $n = 6, 7, 8$, see Figure 1.23.*

(v) *A quiver of type $\mathbb{X}_6$ or of type $\mathbb{X}_7$, see Figure 1.24.*

In general, it is not easy to write a cluster variable in terms of an initial cluster. However, in the case of surface cluster algebras, an explicit formula exists, using so-called snake graphs:

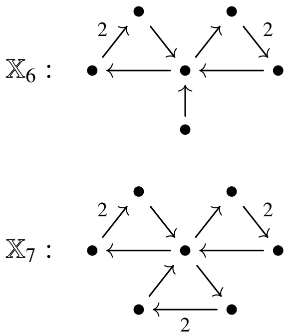


Figure 1.24. Orientations of the two diagrams of type $\mathbb{X}$.

Theorem 1.8.6 (Expansion formula). *Let T be a triangulation of (S, M), assume that T has no self-folded triangles. Let γ be an arc of (S, M) which does not lie in T. Then we have*

$$x_\gamma = \frac{1}{\mathrm{cross}(\gamma)} \sum_{P \in \mathrm{Match}(\mathcal{G}_\gamma)} x(P) y(P),$$

where $x(P)$ is the weight of P, $y(P)$ the height monomial.

We don't have time in these lectures to cover the theory of snake graphs, we refer to [96] or [97] instead.

Chapter 2

Cluster categories

Lecture 4 The combinatorics of cluster algebras and the links with root systems led to the introduction of cluster categories in [26] and [31], independently. These are categories whose (rigid) indecomposable objects are categorical analogues to the cluster variables of a cluster algebra. Cluster categories have a rich representation theory and have been studied intensively in the last two decades. While in [31], the categories are defined from diagonals in a triangulated polygon, the approach in [26] is via orbit categories of bounded derived categories of $\mathrm{mod}(kQ)$ for Q acyclic. Both are examples of additive categorifications in the following sense.

2.1 Categorifications of cluster algebras

Let $\mathcal{A} = \mathcal{A}(Q)$ be the cluster algebra of a cluster quiver Q, with mutable vertices $\{1, 2, \ldots, n\}$ and frozen vertices $\{n + 1, \ldots, m\}$, for $m \geq n$.

Let $\mathcal{F}$ be a k-linear Krull–Remak–Schmidt Frobenius category. (A *Frobenius category* is an exact category with enough projectives and enough injectives and where the classes of projectives and injectives coincide). Let $C := \underline{\mathcal{F}}$ be its stable category: the category C has the same objects as $\mathcal{F}$, the morphisms of C are the morphisms of $\mathcal{F}$, up to factoring through projective-injective objects. As the stable category of a Frobenius category, C is triangulated. Assume that C is Hom-finite, 2-Calabi–Yau (in this case, $\mathcal{F}$ is said to be *stably 2-Calabi–Yau*).

For $\mathcal{F}$ or C to be a categorification of $\mathcal{A}(Q)$, we require that the Grothendieck group of the category is isomorphic to the cluster algebra. We also need a so-called cluster character map from the category to the cluster algebra. Cluster character maps depend on a choice of a cluster tilting object: Fix a basic cluster tilting object $T := T_1 \oplus \cdots \oplus T_n \in C$ of C. An object $M_1 \oplus M_2 \oplus \cdots \oplus M_s$ of $\mathcal{F}$ or C is *basic* if $M_i \not\cong M_j$ for any $1 \leq i \neq j \leq s$.

Let $\widehat{T} := T \oplus P_1 \cdots \oplus P_{m-n} \in \mathcal{F}$ be (basic) cluster tilting in $\mathcal{F}$. Note that for any $M \in \mathcal{F}$, $\mathrm{Ext}^1_{\mathcal{F}}(T, M)$ is a right $\mathrm{End}_C(T)$-module. Then one defines a map $f_T : M \mapsto X^T_m$ to $\mathcal{A}$ as follows:

$$X^T_M := x^{g_{\widehat{T}}(M)} \sum_{\underline{e}} \chi\left(\mathrm{Gr}_{\underline{e}}(\mathrm{Ext}^1_{\mathcal{F}}(T, M))\right) x^{B_T \underline{e}} \in \mathbb{Z}[x_1^{\pm}, \ldots, x_{m+n}^{\pm}],$$

where $g_{\widehat{T}}(M) := [L_0] - [L_1]$ if there is a short exact sequence $0 \to L_1 \to L_0 \to M \to 0$ with $L_i \in \mathrm{add}\,\widehat{T}$ and where $\chi(\mathrm{Gr}_{\underline{e}}(M))$ denotes the Euler characteristic of the quiver Grassmannian of submodules of M of dimension vector $\underline{e}$. We say that the map $f_T : \mathcal{F} \to \mathcal{A}$ is a *cluster character map*.

Definition 2.1.1. With the notation as introduced, the category C is an *additive categorification* of $\mathcal{A}$, if there exists a cluster tilting object $T \in C$, with $\widehat{T} \in \mathcal{F}$, such that the associated cluster character map $f_T : C \to \mathcal{A}$ sends rigid objects to cluster monomials and such that there are bijections

$$\begin{array}{lcl} \{\text{reachable rigid objects}\} & \overset{1:1}{\longleftrightarrow} & \text{cluster monomials} \\ \{\text{reachable indecomposable rigid objects}\} & \overset{1:1}{\longleftrightarrow} & \text{cluster variables} \\ \{\text{cluster tilting objects }\} & \overset{1:1}{\longleftrightarrow} & \text{cluster monomials} \end{array}$$

On the left-hand side, we have to restrict to "reachable" objects as in general, it is not clear whether the exchange graph (under mutation) is connected: if T, T' are two cluster tilting objects, we say that T' is *reachable* from T if there exists a mutation sequence from T to T'. A rigid indecomposable object M is *reachable* from a cluster tilting object T, if M is a direct sum of direct summands of a cluster tilting object T' which is reachable from T. In the sequel, we first build towards the cluster categories as constructed in [26] before we define cluster categories for marked surfaces in Section 2.3 (as in [25]), using Amiot's generalized cluster categories [1].

2.2 Cluster categories from acyclic quivers

We present the definition of a cluster category for an acyclic quiver Q, using its bounded derived category $D^b(kQ)$ (in case Q is acyclic, the module category $\mathrm{mod}(kQ)$ hereditary), as as introduced in [26]. If Q is an orientation of a type A Dynkin diagram, their approach is equivalent to definition in [31]. We recall these categories now.

Definition 2.2.1. Let Q be an acyclic quiver, let $A = kQ$. Let F be the functor $\tau^{-1} \circ [1]$ on the bounded derived category $\mathcal{D} = D^b(A)$ of A.

The *cluster category* $C = C_Q$ *(of* Q*)* is the orbit category $\mathcal{D}/F$ of the functor F in $\mathcal{D}$. Its objects are the F-orbits $\widetilde{M} = (F^i M)_{i\in\mathbb{Z}}$ of objects $M \in \mathcal{D}$ and its morphism are defined as

$$\mathrm{Hom}_C(\widetilde{M}, \widetilde{N}) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(M, F^i N).$$

If Q is an orientation of a Dynkin diagram Δ, we say that $C = C_Q$ is the *cluster category of type* Δ.

We note that the sum in Definition 2.2.1 in the definition of the Hom spaces has only finitely many non-zero summands:

Exercise 16. Let Q be acyclic and $C = C_Q$ be the cluster category of Q. Check that for any two indecomposable objects $M, N \in C$, the sum $\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(M, F^i N)$ in Definition 2.2.1 has at most two non-zero summands.

Remark 2.2.2. Let $\mathcal{D} = D^b(A)$, $F = \tau_{\mathcal{D}}^{-1} \circ [1]_{\mathcal{D}}$ and $C = \mathcal{D}/F$. The cluster category C inherits the shift functor $[1]_C$ and the Auslander–Reiten translation τ_C from the shift functor $[1]_{\mathcal{D}}$ and the Auslander–Reiten translation $\tau_{\mathcal{D}}$ on $\mathcal{D} = D^b(A)$, [26, § 1]. For $M \in \mathcal{D}$, one has

$$\widetilde{M}[1]_C = \widetilde{M[1]_{\mathcal{D}}} \quad \text{and} \quad \tau_C \widetilde{M} = \widetilde{\tau_{\mathcal{D}} M}$$

Furthermore, we have

$$\tau_C = [1]_C, \tag{2.1}$$

since for any indecomposable M, we have $\tau_C \widetilde{M} = \widetilde{\tau_{\mathcal{D}}(FM)} = \widetilde{M[1]_{\mathcal{D}}} = \widetilde{M}[1]_C$.

When it is clear in which category we work, we simply write τ to denote the Auslander–Reiten translation and $[1]$ for the shift functor.

In the following theorem, we denote the standard vector space duality $\mathrm{Hom}_k(-, k)$ by D. To simplify notation, we write M for an object in the cluster category (instead of $\widetilde{M}$).

Theorem 2.2.3. *Let Q be an acyclic quiver. Then the cluster category $C = C_Q$ has the following properties:*

(1) *It is a Hom finite category and has the unique decomposition property,* [26].

(2) *The category C is triangulated,* [77].

(3) *The category C has Serre duality,* $\mathrm{Ext}^1_C(M, N) \cong \mathrm{DHom}_C(N, \tau M)$.

(4) *The category C is 2-Calabi–Yau:*

$$\mathrm{Ext}^1_C(M, N) \cong \mathrm{DExt}^1_C(N, M).$$

Remark 2.2.4. Let Q be an orientation of a Dynkin diagram of type A, D or E, let n be the number of vertices of Q and let C_Q be the associated cluster category. The Auslander–Reiten quiver of $D^b(kQ)$ (vertices: iso classes of indecomposable objects, arrows for irreducible morphisms) has the form $\mathbb{Z} \times Q$ (Happel). The Auslander–Reiten quiver of the module category $\mathrm{mod}(A)$ module category sits inside in $\mathbb{Z} \times Q$ in every degree. Since we take orbits under $F = \tau^{-1}[1]$ when defining C_Q, the indecomposable objects of C_Q arise from the indecomposable objects of the module category of kQ together with n additional indecomposable objects. These are the shifts of the projective-indecomposable kQ-modules. So the Auslander–Reiten quiver of C_Q is obtained from that of $\mathrm{mod}(kQ)$ by adding one slice containing the Dynkin diagram, with appropriate orientation and gluing it to the rest with connecting arrows. See e.g., Section 3.4 in the survey [104]. The Auslander–Reiten quiver either lies on a cylinder with one row of vertices sticking out (types D, E) or on a Möbius strip (type A).

We illustrate the Auslander–Reiten quiver of the cluster category of type A_3, Q : $1 \to 2 \to 3$.

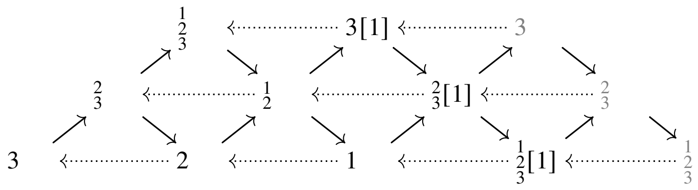


Figure 2.2. The Auslander–Reiten quiver of the cluster category $\mathcal{C}_{A_3}$.

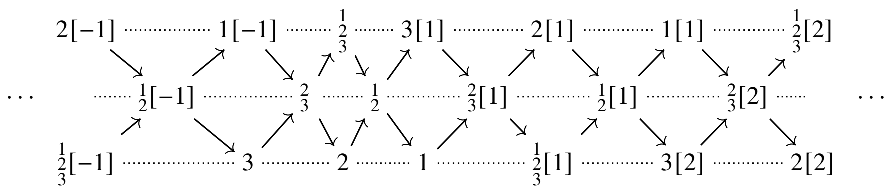


Figure 2.1. The Auslander–Reiten quiver of $D^b(A)$ for $A = k(1 \to 2 \to 3)$.

Example 2.2.5. Consider the Auslander–Reiten quiver of $D^b(A)$ for $A = kQ$, and Q the linear orientation $1 \to 2 \to 3$ of A_3, displayed in Figure 2.1. In the cluster category $\mathcal{C}$, τ is identified with the shift. Therefore, the object 3 gets identified with $F(3) = \tau^{-1}(3[1]) = {\begin{smallmatrix}1\\2\\3\end{smallmatrix}}[2]$. The object ${\begin{smallmatrix}2\\3\end{smallmatrix}}$ gets identified with $F({\begin{smallmatrix}2\\3\end{smallmatrix}}) = \tau^{-1}({\begin{smallmatrix}2\\3\end{smallmatrix}}[1]) = {\begin{smallmatrix}1\\2\end{smallmatrix}}[2]$ and $F({\begin{smallmatrix}1\\2\\3\end{smallmatrix}}) = \tau^{-1}({\begin{smallmatrix}1\\2\\3\end{smallmatrix}}[1]) = 1[2]$: The functor F groups the indecomposable objects of $\mathcal{D}$ into nine orbits, and therefore the cluster category $\mathcal{C}$ has nine indecomposable objects. The Auslander–Reiten quiver of $\mathcal{C}$ is shown in Figure 2.2, with τ indicated by the dotted arrows to the left. In particular, in the cluster category, we only have finitely many (isomorphism classes of) indecomposable objects. The Auslander–Reiten quiver of $\mathcal{C}$ can be embedded in a Möbius strip.

As indicated at the beginning of Chapter 2, cluster categories were introduced as categorical counterparts to cluster algebras. The idea is that cluster variables correspond to indecomposable objects and that clusters correspond to direct sums of indecomposable objects with a fixed number of summands. However, in general there are more indecomposable objects in a cluster category than there are cluster variables in the associated cluster algebra, as non-rigid indecomposable objects do not have counterparts in cluster algebras. We therefore need to restrict to rigid indecomposables to get a bijective correspondence.

Definition 2.2.6. Let $\mathcal{C}$ be the cluster category of an acyclic quiver Q with n vertices.

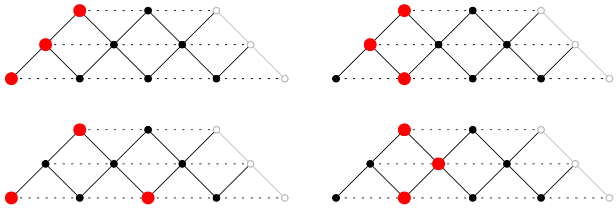


Figure 2.3. Four cluster tilting objects of $\mathcal{C}_{A_3}$, indicated in red.

(1) An object T of a cluster category $\mathcal{C}$ is *rigid* if $\mathrm{Ext}^1_{\mathcal{C}}(T,T) = 0$.

(2) Let T be a rigid object of $\mathcal{C}$. If $T = \oplus_{i=1}^{n} T_i$, with n pairwise non-isomorphic summands $T_1, \dots, T_n$, we say that T is a *cluster tilting object* (or a *maximal rigid object*).

Cluster tilting objects are maximal in the following sense: If T is cluster tilting and if $T_0 \in \mathcal{C}$ is an object such that $T \oplus T_0$ is also rigid, then T_0 has to be a sum of direct summands of T.

Example 2.2.7. We continue with $Q = 1 \to 2 \to 3$, $n = 3$. The following are cluster tilting objects of $\mathcal{C}$:

$$M_1 := 3 \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}, \quad M_2 := 3 \oplus \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus 1, \quad M_3 := 2 \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}, \quad M_4 := 2 \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}.$$

These four are shown as red vertices in Figure 2.3 (using the Auslander–Reiten quiver from Figure 2.2). Any other cluster tilting object is a shift $M_i[j]$ (or a τ-translate $\tau^j(M_i)$) of one of the above. In total, one gets 14 cluster tilting objects, listed as τ-orbits of cluster-tilting objects:

$$\begin{aligned} &M_1, \tau(M_1), \tau^2(M_1), \tau^3(M_1), \tau^4(M_1), \tau^5(M_5), \\ &M_3, \tau(M_3), \tau^2(M_3), \\ &M_2, \tau(M_2), \tau^2(M_2), \\ &M_4, \tau(M_4). \end{aligned}$$

We note that cluster tilting objects in $\mathcal{C}_Q$ are analogues of clusters in the cluster algebra $\mathcal{A}(Q)$. The mutation at a variable of a cluster corresponds to an exchange of summands of cluster tilting objects ([26, Section 6]).

Theorem 2.2.8. *Let Q be an acyclic quiver, $\mathcal{C} = \mathcal{C}_Q$. Let T be a cluster tilting object of $\mathcal{C}$, M be an indecomposable direct summand of T and define $\overline{T}$ by $T = \overline{T} \oplus M$. Then there exists a unique indecomposable object $M' \not\cong M$ such that $\overline{T} \oplus M'$ is also cluster* *tilting. Moreover, there are two unique triangles in* C *linking* M *and* M'*:*

$$M' \longrightarrow \bigoplus_{i\in I} B_i \longrightarrow M \longrightarrow M'[1],$$

$$M \longrightarrow \bigoplus_{j} \in I' B'_{j'} \longrightarrow M' \longrightarrow M[1],$$

where the objects B_i, B'_j *are indecomposable summands of* T *and where* I*,* I' *are the corresponding index sets.*

There are also maps between (rigid) indecomposable objects and cluster variables: In [30], an explicit map $M \mapsto x_M$ from indecomposable objects to cluster variables is given. This map is surjective ([32]) and the article [29] gives a map from cluster variables to rigid indecomposable objects. Combining these, as shown in the appendix of [29] (co-authored with Caldero and Keller), one gets a bijection between rigid indecomposable objects of C_Q and cluster variables of $\mathcal{A}(Q)$. In turn, one gets a bijection between cluster tilted objects and clusters. This is summarised in part (1) of the theorem below. In addition, by [29, Theorem 6.1], the exchange formula for cluster variables can be given in terms of the two complements (part (2) of the theorem).

Theorem 2.2.9. *Let* Q *be an acyclic quiver with* n *vertices,* $C = C_Q$ *the associated cluster category and* $\mathcal{A} = \mathcal{A}(\mathbf{x}, Q)$ *its cluster algebra of* Q *with* $\mathbf{x} = (x_1, \dots, x_n)$*.*

(1) *There are bijections*

$$\begin{array}{rcl} \{\textit{indecomposable rigid objects in } C\} & \longleftrightarrow & \{\textit{cluster variables in } \mathcal{A}\}; \\ M & \longrightarrow & x_M, \end{array}$$

$$\begin{array}{rcl} \{\textit{cluster tilting objects in } C\} & \longleftrightarrow & \{\textit{clusters in } \mathcal{A}\}; \\ T = \bigoplus_{i=1}^{n} T_i & \longrightarrow & x_T = (x_{T_1}, \dots, x_{T_n}), \end{array}$$

(2) *For any two rigid indecomposable objects* $M, N \in C$ *we have*

$$\dim \operatorname{Ext}^1_C(M, N) = 1 \quad \Longleftrightarrow \quad x_M \textit{ and } x_N \textit{ form an exchange pair.}$$

Moreover, in this situation, the exchange relation is given by

$$x_M \cdot x_N := (*_1) \prod_{i\in I} x_{B_i} + (*_2) \prod_{j\in I'} x_{B'_j},$$

where $(*_1), (*_2)$ *are some coefficients and where the* B_i*,* B'_j *are indecomposable summands of* T *(as in Theorem 2.2.8).*

Using Theorems 2.2.8 and 2.2.9, the following definition makes sense: Let T be a cluster tilting object, M an indecomposable summand of T, $T = \overline{T} \oplus M$ and let M' be the other complement to $\overline{T}$. Set $T' := \overline{T} \oplus M'$. We say that the *mutation of T at M* is the indecomposable object M' (and vice versa):

$$\mu_M(T) = \overline{T} \oplus M' \quad \text{and} \quad \mu_{M'}(T') = \overline{T} \oplus M.$$

In the next section, we will show how cluster categories arise from triangulations of marked surfaces. Quivers associated to triangulations often contain oriented cycles and so the methods from Section 2.2 cannot be applied. It becomes necessary to work with relations (via quivers with potentials) and use Amiot's generalized cluster categories.

2.3 The cluster category of a marked surface

The first approach to cluster categories via surfaces was given by Caldero, Chapoton and Schiffler in [31] where the authors use the diagonals in a convex $n+3$-gon and rotations between them to define the Auslander–Reiten quiver of a cluster category in type A_n. Since then, many geometric models for cluster categories and related categories have appeared. Cluster categories in type D_n were associated to punctured disks by Schiffler in [107], to m-cluster categories in type A_n in [11], in type D_n in [10], to tube categories in [14]. The work [25] of Brüstle–Zhang uses Amiot's generalized cluster categories ([1]) to define cluster categories for marked surfaces without punctures and [102] associates them to surfaces with punctures. The goal of this section is to present their approach.

Let T be a triangulation of a marked surface (S, M). For simplicity, we assume that T has no self-folded triangles (this will be useful when discussing the potentials for surface triangulations). Let Q_T be the quiver of T (Definition 1.6.2). Its vertices are the arcs of the triangulation and there is an arrow $i \to j$ if j follows i clockwise inside a common triangle.

In general, a triangulated surface is obtained from gluing together triangles, digons with one puncture and monogons with two punctures (see Figure 1.19). Since we have no self-folded triangles, only the first types of puzzle pieces (triangles) are used. Some of the sides of these puzzle pieces become boundary segments of the surface. Accordingly, the quiver Q_T is obtained from gluing together arrows and clockwise oriented 3-cycles (for the internal triangles). The resulting quiver may also contain anticlockwise circles given by edges incident with a common puncture.

The pictures on the left hand in Figure 2.4 shows the quiver of triangulations (in red).

We extend the quiver Q_T by adding arrows between boundary segments incident with a common marked point and between arcs of T and "neighboured" arcs:

Definition 2.3.1. Let $S = (S, M)$ be a marked surface. Let T be a triangulation of S with n arcs, labelled $1, 2, \dots, n$. Label the boundary segments by $n+1, \dots, m$, for some $m > n$. The *extended quiver $\widetilde{Q}_T$ of the triangulation T* is defined as follows: The

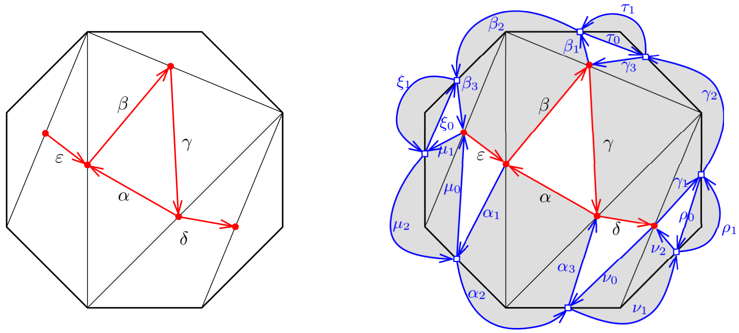


Figure 2.4. The quivers Q_T and $\widetilde{Q}_T$ of a triangulation T of an octagon.

vertices of $\widetilde{Q}_T$ are the arcs of T and the boundary segments of (S, M). We draw an arrow $i \longrightarrow j$ (with $i, j \in \{1, 2, \ldots, m\}$) in the following situations:

- j follows i clockwise inside a common triangle.
- i, j are both boundary segments with a common endpoint and j follows i anti-clockwise when going around outside the boundary of S.

At the end, remove a maximal collection of 2-cycles between vertices corresponding to arcs of T.

An example of the extended quiver is given on the right-hand side of Figure 2.4. The additional vertices (for the boundary segments) are drawn as blue boxes, the additional arrows are blue. Note that in contrast to Q_T, the extended quiver $\widetilde{Q}_T$ may have two-cycles between vertices corresponding to boundary segments. See the right hand of Figure 2.4 for an example with three two-cycles at the boundary.

The resulting quiver $\widetilde{Q}_T$ is an example a dimer quiver with boundary see Definition 2.3.2 below: it is a quiver with faces. Informally speaking, such a quiver it is built by gluing faces along common arrows, in a consistent way. Dimer quivers (or dimer models) with boundary were introduced in [9, Section 3] as a tool towards a surface approach to Grassmannian cluster categories.

Definition 2.3.2. A (finite, connected) *dimer quiver (with boundary)* is a finite quiver $Q = (Q_0, Q_1, Q_2)$ with faces which contains no loops where the faces Q_2 are a disjoint union $Q_2 = Q_2^+ \cup Q_2^-$ of positively, respectively negatively oriented faces, satisfying the conditions

(i) every arrow in Q_1 either has face multiplicity two[1] or face multiplicity one. In the former case, it is called an *internal arrow*, in the latter case, it is a *boundary arrow*.

[1]The face multiplicity of an arrow is the number of times an arrow appears in the boundaries of faces in Q_2.

(ii) The incidence graph at each vertex of Q is connected[2].

Let Q be a dimer quiver with boundary. Any vertex of Q incident with only boundary arrows is called a *boundary vertex*. All other vertices are called *internal*.
If an oriented cycle σ of Q does not contain any cyclic subpaths, it is called a *unit cycle*.

By condition (i) of Definition 2.3.2, the incidence graph at boundary vertices of Q is a line, and it is a cycle at internal vertices.

Exercise 17. Let T be a triangulation of a surface (S, M). Check that the extended quiver $\widetilde{Q}_T$ is a dimer quiver with boundary.

Let Q be a dimer quiver with boundary. It can be embedded in a surface with boundary. Each face F of Q can be viewed as a polygon whose edges are arrows of ∂F, forming a positive or a negative cycle. By gluing together edges of the polygons labelled by the same arrows, compatibly with the orientation of the arrows, we obtain an oriented surface which we denote by $|Q|$ ([9, Section 3]). Each component of the boundary of the surface is identified with an unoriented cycle formed by boundary arrows of Q.

Definition 2.3.3. A *potential W for Q* is a (possibly infinite) formal linear combination of cyclic paths in the (completed) path algebra $k[\![Q]\!]$. If W is a potential for Q, the pair (Q, W) is a *quiver with potential*, in short a QP.

Example 2.3.4. Quivers of surface triangulations and dimer quivers (with or without boundary) are important families of QPs, they come with a natural potential given by the orientation of the surface. (The extended quivers of triangulations belong to both families). Quivers with potentials for triangulated surfaces have been first studied in [85], see Definition 23 in that article.

Definition 2.3.5. (1) Let T be a triangulation of (S, M) and Q_T its quiver. We denote by Q_2^+ the faces given by the positive three-cycles corresponding to internal triangles and by Q_2^- all the other faces. We use these unit cycles to define a potential $W_T = W_{Q_T}$ for Q_T. Let C^+ be the set of all positive three-cycles in Q_T, up to cyclic equivalence and let C^- be the set of negative unit cycles of the boundaries of faces in Q_2^-. We define the *natural potential of T* to be

$$W_T = \sum_{\sigma \in C^+} \sigma - \sum_{\sigma \in C^-} \sigma.$$

If (S, M) has no punctures, the quiver Q_T only has positive three-cycles and the potential is $W_T = \sum_{\sigma \in C^+} \sigma$.

[2]The incidence graph of Q at a vertex $i \in Q_0$ is an unoriented graph which has as vertices the arrows incident with i. Its edges correspond to 2-cycles going through i.

(2) More generally, let Q be a dimer quiver and $|Q|$ the surface it defines (Q may be the extended quiver $\widetilde{Q}_T$ of a surface triangulation T). Let C^+ be the set of positive unit cycles of Q and C^- the set of negative unit cycles, all up to cyclic equivalence - the two sets are given by the boundary cycles of the faces of Q. The *(natural) potential of* Q is

$$W_Q := \sum_{\sigma \in C^+} \sigma - \sum_{\sigma \in C^-} \sigma. \tag{2.2}$$

Recall that we assume that T has no self-folded triangles. In fact, if a triangulation contains self-folded triangles, the description of a potential is more complicated.

Example 2.3.6. We consider the triangulation of an octagon as in Figure 2.4. There is exactly one 3-cycle (corresponding to the inner triangle in the triangulation). The extended quiver $\widetilde{Q}_T$ with arrows to the boundary segments and between them has six positive unit 3-cycles. They correspond to the six triangles in T. It has eight negative unit cycles, they appear around the eight vertices of the octagon: three 2-cycles, two five-cycles (one containing ε and β, one containing γ), one 4-cycle (containing α) and two 3-cycles. In the figure, they are shaded in grey. So the potentials from Definition 2.3.5 are:

$$\begin{aligned} W_T &= \alpha\beta\gamma, \\ W_{\widetilde{Q}_T} &= \text{“cycles from white faces”} - \text{“cycles from grey faces”}. \end{aligned}$$

Derksen, Weyman and Zelevinsky introduced mutation of quivers with potentials [40]. Labardini–Fragoso has shown ([85, Theorem 30]) that for surface triangulations (without self-folded triangles) with the above natural potential, mutation of QP corresponds to flipping arcs, providing a result analogous to the fact that mutation of quivers associated with surface triangulations corresponds to the flip (Proposition 1.8.3 (1)):

Theorem 2.3.7. *Let T be a triangulation of a surface (S, M) and T' obtained from T by flipping an arc of T, then the QP mutation of (Q_T, W_T) is the quiver with potential $(Q_{T'}, W_{T'})$.*

Quivers with potentials are a key ingredient towards cluster categories. First we define algebras from them.

Definition 2.3.8. Let (Q, W) be the quiver with potential of a triangulation or a dimer quiver (potentially with boundary) with the potential as in (2.2).

(1) Let a be an arrow of Q. The *cyclic derivative* ∂_a *with respect to* a sends any unit cycle $a_1 a_2 \cdots a_d$ to the sum $\sum_{k=1}^{d} \delta_{a_k,a} a_{k+1} \cdots a_d a_1 \cdots a_{k-1}$. This is linearly extended to the potential W, we write $\partial_a W$ for the cyclic derivative of W w.r.t. a.

(2) If (Q, W) is the quiver with potential (Q_T, W_T) of a surface triangulation, let $I(Q, W) := \langle \{\partial_a W \mid a \in Q_1\} \rangle$ be the ideal generated by all cyclic derivatives of W. If Q is a dimer quiver (with boundary), let $I(Q, W)$ be the closure of the ideal generated by the cyclic derivatives $\partial_a W$ of all internal arrows of Q.

The *frozen Jacobian algebra* $J(Q,W)$ *of* (Q,W) is the quotient of the completed path algebra $k[\![Q]\!]$ by the closure of the ideal generated $I(Q,W)$:

$$J(Q,W) = k[\![Q]\!]/I(Q,W)$$

Remark 2.3.9. The frozen Jacobian algebra is not always finite dimensional. If it is finite dimensional, we do not have to take the completion. By [34, Theorem 5.7], this is in particular true for quivers with potentials from arbitrary triangulations of surfaces (generalising [85, Theorem 36]).

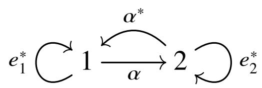
For the rest of the section we assume that (S,M) does not have any punctures. Then we can work in the setting of [25]. Such surfaces have no self-folded triangles, and we can use the natural potential given in Example 2.3.4. To work with punctured surfaces requires dealing with more involved potentials and with Jacobian algebras which are not gentle. This has been achieved In [102] where cluster categories are defined for punctured cases.

We associate another algebra to a quiver with potential, the so-called Ginzburg dg-algebra, following exposition given in [92]. See also the survey [2] for details.

Let (Q,W) be a quiver with potential. From Q, we define a graded quiver $\overline{Q}$. It has the same vertices as Q. The arrows of $\overline{Q}$ come in three types:

- The degree 0 arrows of $\overline{Q}$ are the arrows of Q;
- for every arrow $\alpha : i \to j$ of Q, one draws an arrow $\alpha^* : j \to i$ in $\overline{Q}$, it is of degree -1;
- For every vertex $i \in Q_0$, the quiver $\overline{Q}$ has a loop e_i^* at i, it is of degree -2.

Example 2.3.10. For Q a quiver of type A_2 with arrow $\alpha : 1 \to 2$, the graded quiver $\overline{Q}$ looks as follows:



Definition 2.3.11. Let (Q,W) be a quiver with potential, let $\overline{Q}$ be the graded quiver of Q. The *Ginzburg dg-algebra* has as underlying graded algebra the path algebra of $\overline{Q}$. Its differential d is defined as follows:

- $d(\alpha) = 0$ for every $\alpha \in Q_1$ and $d(e_i) = 0$ for every vertex i.
- $d(\alpha^*) = \partial_\alpha W$ for every α in Q_1.
- $d(e_i^*) = e_i(\sum_{\alpha\in Q_1} \alpha\alpha^* - \alpha^*\alpha)e_i$ for every vertex i.

The Ginzburg dg algebra is non-zero only in negative degrees and the cohomology $H^0(\widehat{\Gamma(Q,W)})$ of the completed dg algebra (completion w.r.t. the arrow ideal) is isomorphic to the Jacobian algebra $J(Q,W)$, see [2, Section 3].

From now, let (Q,W) be the quiver with potential of a triangulation of a surface without punctures. In that case, the Jacobian algebra $J(Q,W)$ is finite dimensional, and we do not need to take the completions. Let $\Gamma = \Gamma(Q, W)$ be the associated Ginzburg dg-algebra. Let $D\Gamma$ be the derived category of Γ and $D^b\Gamma$ the bounded derived category, [78]. The *perfect derived category* $\operatorname{per}\Gamma$ is the smallest triangulated subcategory of $D\Gamma$ which contains Γ and which is stable under taking direct summands. By a result of Keller, [79, Section 6], the bounded derived category $D^b\Gamma$ is a subcategory of $\operatorname{per}\Gamma$. The *(Verdier) localisation* $\operatorname{per}\Gamma/D^b\Gamma$ *of* $\operatorname{per}\Gamma$ is the category obtained by formally inverting the morphisms of $\operatorname{per}\Gamma$ whose cone belongs to $D^b\Gamma$ (see for example [83]). Now we have everything together for the definition of the (generalized) cluster category of (Q, W) of Amiot, [1, Section 3].

Definition 2.3.12. Let (Q, W) be a quiver with potential such that $J(Q, W)$ is finite dimensional. The *(generalized) cluster category* $\mathcal{C}_{(Q,W)}$ *associated to* (Q, W) is the (Verdier) localisation $\operatorname{per}\Gamma/D^b\Gamma$.

This definition can be applied in particular to quivers with potential arising from surface triangulations where (S, M) has no punctures, since in this case, the Jacobian algebra is always finite dimensional (Remark 2.3.9).

Note that while the definition of a cluster category $\mathcal{C}_Q$ from [26] requires Q to be acyclic, i.e., only works for algebra of global dimension at most 1, Amiot's construction allows algebras of global dimension at most 2. It recovers the category $\mathcal{C}_Q$, as Amiot showed in [1]:

Theorem 2.3.13. *The (generalized) cluster category $\mathcal{C}_{(Q,W)}$ is Hom-finite and 2-Calabi–Yau. If Q is acyclic, then $W = 0$ and the triangulated category $\mathcal{C}_{(Q,0)}$ is equivalent to the acyclic cluster category $\mathcal{C}_Q = D^b(Q)/\tau^{-1}$* [1].

Remark 2.3.14. In fact, Amiot showed that the image of $\Gamma(Q, W)$ in $\mathcal{C}_{(Q,W)}$ is a cluster tilting object whose endomorphism algebra is isomorphic to $J(Q, W)$.

We have seen that the cluster algebra of a surface triangulation does not depend on the chosen triangulation (Proposition 1.8.3 (2)). An analogous statement is true for the cluster category of a triangulation. Keller and Yang showed in [81, Section 3] that mutation induces a triangle equivalence for generalized cluster categories. For surface triangulations, their result becomes:

Theorem 2.3.15. *Let (S, M) be a marked surface without punctures. Let T be a triangulation of (S, M), and let T' be a triangulation obtained from T by flipping a diagonal. Then there is a triangle equivalence*

$$\mathcal{C}_{(Q_T, W_T)} \cong \mathcal{C}_{(Q_{T'}, W_{T'})} \,.$$

As any two triangulations of a given surface are related by a (finite) sequence of flips (Theorem 1.3.5), Theorem 2.3.15 shows that the cluster category $\mathcal{C}_{(Q_T, W_T)}$ is independent of the choice of the triangulation of (S, M) (up to triangle equivalence).

Therefore, the following definition makes sense:

Definition 2.3.16. Let (S, M) be a marked surface without punctures. The *cluster category of* (S, M) is defined to be $\mathcal{C}_{(Q_T, W_T)}$ where (Q_T, W_T) is the quiver with potential of a triangulation of (S, M).

In this generality, the categories $\mathcal{C}_{(S,M)}$ have been introduced by Brüstle–Zhang in [25]. We recall their main results below. To fix the notation, let T be a triangulation of a surface (S, M), let (Q, W) be its quiver with potential. Let T be the image of $\Gamma(Q, W)$ under per $\Gamma(Q, W) \to \mathcal{C}_{(Q,W)}$. By Remark 2.3.14, the object T is a cluster tilting object. Every arc γ in (S, M) which is not part of T is associated with an indecomposable $J(Q, W)$-module which we denote by $I(\gamma)$, [3, Proposition 4.2]. We write M_γ to denote the unique (up to isomorphism) indecomposable object of $\mathcal{C}_{(S,M)}$ with $\mathrm{Hom}_{\mathcal{C}_{(S,M)}}(T, M_\gamma[1]) \cong I(\gamma)$. For any arc γ_k of the triangulation T, we write M_{γ_k} to denote the corresponding summand of T. Note that an object M of $\mathcal{C}_{(S,M)}$ is said to be *exceptional* if it is indecomposable and rigid.

Theorem 2.3.17. *Let $\mathcal{C}_{(S,M)}$ be the cluster category of (S, M).*

- *The map $\gamma \mapsto M_\gamma$ is a bijection between the arcs of (S, M) and the (isomorphism classes of) exceptional objects of $\mathcal{C}_{(S,M)}$.*
- *For any two exceptional objects M_α and M_β of $\mathcal{C}_{(S,M)}$, we have*

$$\mathrm{Ext}^1_{\mathcal{C}_{(S,M)}}(M_\alpha, M_\beta) = 0 \Longleftrightarrow \alpha \text{ and } \beta \text{ do not cross.}$$

- *The shift functor of $\mathcal{C}_{(S,M)}$ acts on the arcs of (S, M) by moving both endpoints clockwise along the boundary to the next marked points.*

Remark 2.3.18. Summarising, if (S, M) is a surface without punctures, we have correspondences:

$\mathcal{A}_{(S,M)}$	(S, M)	$\mathcal{C}_{(S,M)}$
x_α cluster variable	arcs $\alpha \in (S, M)$	M_α rigid indecomposable
clusters	triangulations	cluster tilting objects
mutation	flip	mutation
x_α and x_β compatible	α, β do not cross	$\mathrm{Ext}^1_{\mathcal{C}_{(S,M)}}(M_\alpha, M_\beta) = 0$
(x_α, x_β) exchange pair	$\exists\, T \neq T'$ with $T \setminus \alpha \cup \beta = T'$	M_α and M_β form an exchange pair

Remark 2.3.19. The cluster category $\mathcal{C}_{(S,M)}$ is an additive categorification of the cluster algebra $\mathcal{A}(Q)$, where $Q = Q_T$ is the quiver of a triangulation of (S, M), see Definition 2.1.1.

References

[1] C. Amiot, Cluster categories for algebras of global dimension 2 and quivers with potential. *Ann. Inst. Fourier (Grenoble)* **59** (2009), 2525–2590

[2] C. Amiot, On generalized cluster categories. In *Representations of algebras and related topics*, pp. 1–53, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, URL https://doi.org/10.4171/101-1/1

[3] I. Assem, T. Brüstle, G. Charbonneau-Jodoin and P.-G. Plamondon, Gentle algebras arising from surface triangulations. *Algebra Number Theory* **4** (2010), 201–229

[4] I. Assem, G. Dupont, R. Schiffler and D. Smith, Friezes, strings and cluster variables. *Glasg. Math. J.* **54** (2012), 27–60

[5] I. Assem, C. Reutenauer and D. Smith, Friezes. *Adv. Math.* **225** (2010), 3134–3165

[6] I. Assem, D. Simson and A. Skowroński, *Elements of the representation theory of associative algebras. Vol. 1*. London Mathematical Society Student Texts 65, Cambridge University Press, Cambridge, 2006

[7] C. A. Athanasiadis and E. Tzanaki, Shellability and higher Cohen-Macaulay connectivity of generalized cluster complexes. *Israel J. Math.* **167** (2008), 177–191

[8] M. Auslander, I. Reiten and S. O. Smalø, *Representation theory of Artin algebras*. Cambridge Studies in Advanced Mathematics 36, Cambridge University Press, Cambridge, 1995

[9] K. Baur, A. D. King and B. R. Marsh, Dimer models and cluster categories of Grassmannians. *Proc. Lond. Math. Soc. (3)* **113** (2016), 213–260

[10] K. Baur and B. R. Marsh, A geometric description of the m-cluster categories of type D_n. *Int. Math. Res. Not. IMRN* (2007), Art. ID rnm011, 19

[11] K. Baur and B. R. Marsh, A geometric description of m-cluster categories. *Trans. Amer. Math. Soc.* **360** (2008), 5789–5803

[12] K. Baur and B. R. Marsh, Frieze patterns for punctured discs. *J. Algebraic Combin.* **30** (2009), 349–379

[13] K. Baur and B. R. Marsh, Categorification of a frieze pattern determinant. *J. Combin. Theory Ser. A* **119** (2012), 1110–1122

[14] K. Baur and B. R. Marsh, A geometric model of tube categories. *J. Algebra* **362** (2012), 178–191

[15] V. Bazier-Matte and R. Schiffler, Knot theory and cluster algebras. *Adv. Math.* **408** (2022), Paper No. 108609, 45

[16] F. Benini, D. S. Park and P. Zhao, Cluster algebras from dualities of 2d $\mathcal{N} = (2,2)$ quiver gauge theories. *Comm. Math. Phys.* **340** (2015), 47–104

[17] D. J. Benson, *Representations and cohomology. I*. Second edn., Cambridge Studies in Advanced Mathematics 30, Cambridge University Press, Cambridge, 1998

[18] A. Berenstein, S. Fomin and A. Zelevinsky, Cluster algebras. III. Upper bounds and double Bruhat cells. *Duke Math. J.* **126** (2005), 1–52

[19] A. Berenstein and A. Zelevinsky, Quantum cluster algebras. *Adv. Math.* **195** (2005), 405–455

[20] R. Bocklandt, Consistency conditions for dimer models. *Glasg. Math. J.* **54** (2012), 429–447

[21] K. Bongartz, Critical simply connected algebras. *Manuscripta Math.* **46** (1984), 117–136

[22] L. Bossinger, B. Frías-Medina, T. Magee and A. Nájera Chávez, Toric degenerations of cluster varieties and cluster duality. *Compos. Math.* **156** (2020), 2149–2206

[23] M. Brion, Representations of quivers. In *Geometric methods in representation theory. I*, pp. 103–144, Sémin. Congr. 24, Soc. Math. France, Paris, 2012

[24] N. Broomhead, Dimer models and Calabi-Yau algebras. *Mem. Amer. Math. Soc.* **215** (2012), viii+86

[25] T. Brüstle and J. Zhang, On the cluster category of a marked surface without punctures. *Algebra Number Theory* **5** (2011), 529–566

[26] A. Buan, B. Marsh, M. Reineke, I. Reiten and G. Todorov, Tilting theory and cluster combinatorics. *Advances in mathematics* **204** (2006), 572–618

[27] A. B. Buan and B. Marsh, Cluster-tilting theory. In *Trends in representation theory of algebras and related topics*, pp. 1–30, Contemp. Math. 406, Amer. Math. Soc., Providence, RI, 2006, URL https://doi.org/10.1090/conm/406/07651

[28] A. B. Buan, B. R. Marsh and I. Reiten, Cluster mutation via quiver representations. *Comment. Math. Helv.* **83** (2008), 143–177

[29] A. B. Buan, B. R. Marsh, I. Reiten and G. Todorov, Clusters and seeds in acyclic cluster algebras. *Proc. Amer. Math. Soc.* **135** (2007), 3049–3060

[30] P. Caldero and F. Chapoton, Cluster algebras as Hall algebras of quiver representations. *Comment. Math. Helv.* **81** (2006), 595–616

[31] P. Caldero, F. Chapoton and R. Schiffler, Quivers with relations arising from clusters (A_n case). *Transactions of the American Mathematical Society* **358** (2006), 1347–1364

[32] P. Caldero and B. Keller, From triangulated categories to cluster algebras. II. *Ann. Sci. École Norm. Sup. (4)* **39** (2006), 983–1009

[33] I. Canakcı and R. Schiffler, Snake graph calculus and cluster algebras from surfaces. *J. Algebra* **382** (2013), 240–281

[34] G. Cerulli Irelli and D. Labardini-Fragoso, Quivers with potentials associated to triangulated surfaces, Part III: tagged triangulations and cluster monomials. *Compos. Math.* **148** (2012), 1833–1866

[35] F. Chapoton, S. Fomin and A. Zelevinsky, Polytopal realizations of generalized associahedra. *Canad. Math. Bull.* **45** (2002), 537–566

[36] J. Conway and H. Coxeter, Triangulated polygons and frieze patterns. *The Mathematical Gazette* **57** (1973), 87–94

[37] J. Conway and H. Coxeter, Triangulated polygons and frieze patterns. *The Mathematical Gazette* **57** (1973), 175–183

[38] M. Cuntz and I. Heckenberger, Reflection groupoids of rank two and cluster algebras of type *A*. *J. Combin. Theory Ser. A* **118** (2011), 1350–1363

[39] B. Davison, Consistency conditions for brane tilings. *J. Algebra* **338** (2011), 1–23

[40] H. Derksen, J. Weyman and A. Zelevinsky, Quivers with potentials and their representations. I. Mutations. *Selecta Math. (N.S.)* **14** (2008), 59–119

[41] S.-P. Eu and T.-S. Fu, The cyclic sieving phenomenon for faces of generalized cluster complexes. *Adv. in Appl. Math.* **40** (2008), 350–376

[42] B. Farb and D. Margalit, *A primer on mapping class groups*. Princeton Mathematical Series 49, Princeton University Press, Princeton, NJ, 2012

[43] A. Felikson, M. Shapiro and P. Tumarkin, Skew-symmetric cluster algebras of finite mutation type. *Journal of the European Mathematical Society* **14** (2012), 1135–1180

[44] V. V. Fock and A. B. Goncharov, Cluster $\mathcal{X}$-varieties, amalgamation, and Poisson-Lie groups. In *Algebraic geometry and number theory*, pp. 27–68, Progr. Math. 253, Birkhäuser Boston, Boston, MA, 2006, URL https://doi.org/10.1007/978-0-8176-4532-8_2

[45] V. V. Fock and A. B. Goncharov, Dual Teichmüller and lamination spaces. In *Handbook of Teichmüller theory. Vol. I*, pp. 647–684, IRMA Lect. Math. Theor. Phys. 11, Eur. Math. Soc., Zürich, 2007, URL https://doi.org/10.4171/029-1/16

[46] V. V. Fock and A. B. Goncharov, Cluster ensembles, quantization and the dilogarithm. *Ann. Sci. Éc. Norm. Supér. (4)* **42** (2009), 865–930

[47] V. V. Fock and A. B. Goncharov, The quantum dilogarithm and representations of quantum cluster varieties. *Invent. Math.* **175** (2009), 223–286

[48] V. V. Fock and A. Marshakov, Loop groups, clusters, dimers and integrable systems. In *Geometry and quantization of moduli spaces*, pp. 1–66, Adv. Courses Math. CRM Barcelona, Birkhäuser/Springer, Cham, 2016

[49] S. Fomin and N. Reading, Generalized cluster complexes and Coxeter combinatorics. *Int. Math. Res. Not.* (2005), 2709–2757

[50] S. Fomin, M. Shapiro and D. Thurston, Cluster algebras and triangulated surfaces. Part I: Cluster complexes. *Acta Mathematica* **201** (2008), 83–146

[51] S. Fomin, L. Williams and A. Zelevinsky, Introduction to cluster algebras. chapters 1-3. *arXiv preprint arXiv:1608.05735* (2024)

[52] S. Fomin and A. Zelevinsky, Cluster algebras I: Foundations. *Journal of the American Mathematical Society* **15** (2002), 497–529

[53] S. Fomin and A. Zelevinsky, Cluster algebras. II. Finite type classification. *Invent. Math.* **154** (2003), 63–121

[54] S. Fomin and A. Zelevinsky, Y-systems and generalized associahedra. *Ann. of Math. (2)* **158** (2003), 977–1018

[55] S. Franco, Bipartite field theories: from D-brane probes to scattering amplitudes. *J. High Energy Phys.* (2012), 141, front matter + 48

[56] S. Franco, D. Galloni and A. Mariotti, Bipartite field theories, cluster algebras and the Grassmannian. *J. Phys. A* **47** (2014), 474004, 30

[57] S. Franco, A. Hanany, D. Vegh, B. Wecht and K. D. Kennaway, Brane dimers and quiver gauge theories. *J. High Energy Phys.* (2006), 096, 48

[58] C. Geiss, B. Leclerc and J. Schröer, Auslander algebras and initial seeds for cluster algebras. *J. Lond. Math. Soc. (2)* **75** (2007), 718–740

[59] C. Geiss, B. Leclerc and J. Schröer, Partial flag varieties and preprojective algebras. *Ann. Inst. Fourier (Grenoble)* **58** (2008), 825–876

[60] M. Gekhtman, M. Shapiro, S. Tabachnikov and A. Vainshtein, Integrable cluster dynamics of directed networks and pentagram maps. *Adv. Math.* **300** (2016), 390–450

[61] M. Gekhtman, M. Shapiro and A. Vainshtein, Cluster algebras and Poisson geometry. *Mosc. Math. J.* **3** (2003), 899–934, 1199

[62] M. Gekhtman, M. Shapiro and A. Vainshtein, Cluster algebras and Weil-Petersson forms. *Duke Math. J.* **127** (2005), 291–311

[63] M. Glick and D. Rupel, Introduction to cluster algebras. In *Symmetries and integrability of difference equations*, pp. 325–357, CRM Ser. Math. Phys., Springer, Cham, 2017

[64] A. B. Goncharov and R. Kenyon, Dimers and cluster integrable systems. *Ann. Sci. Éc. Norm. Supér. (4)* **46** (2013), 747–813

[65] J. E. Grabowski and S. Gratz, Cluster algebras of infinite rank. *J. Lond. Math. Soc. (2)* **89** (2014), 337–363

[66] M. Gross, P. Hacking, S. Keel and M. Kontsevich, Canonical bases for cluster algebras. *J. Amer. Math. Soc.* **31** (2018), 497–608

[67] L. Guo, On tropical friezes associated with Dynkin diagrams. *Int. Math. Res. Not. IMRN* (2013), 4243–4284

[68] P. Hacking and S. Keel, Mirror symmetry and cluster algebras. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures*, pp. 671–697, World Sci. Publ., Hackensack, NJ, 2018

[69] S. Hassoun, D. Langford and F. Langlois, La structure d'algèbre amassée des grassmaniennes gr(k,n). 2018, URL https://usherbrooke.scholaris.ca/server/api/core/bitstreams/d1a879b7-ec14-40df-a3ce-c739d2b108f5/content

[70] A. Hatcher, On triangulations of surfaces. *Topology Appl.* **40** (1991), 189–194

[71] K. Hikami and R. Inoue, Braids, complex volume and cluster algebras. *Algebr. Geom. Topol.* **15** (2015), 2175–2194

[72] T. Holm and P. Jø rgensen, On a cluster category of infinite Dynkin type, and the relation to triangulations of the infinity-gon. *Math. Z.* **270** (2012), 277–295

[73] T. Holm and P. Jø rgensen, A p-angulated generalisation of Conway and Coxeter's theorem on frieze patterns. *Int. Math. Res. Not. IMRN* (2020), 71–90

[74] K. Igusa and G. Todorov, Cluster categories coming from cyclic posets. *Comm. Algebra* **43** (2015), 4367–4402

[75] D. Kaufman, Mutation invariant functions on cluster ensembles. *J. Pure Appl. Algebra* **228** (2024), Paper No. 107495, 25

[76] R. Kedem, Q-systems as cluster algebras. *J. Phys. A* **41** (2008), 194011, 14

[77] B. Keller, On triangulated orbit categories. *Doc. Math.* **10** (2005), 551–581

[78] B. Keller, On differential graded categories. In *International Congress of Mathematicians. Vol. II*, pp. 151–190, Eur. Math. Soc., Zürich, 2006

[79] B. Keller, Deformed Calabi-Yau completions. *J. Reine Angew. Math.* **654** (2011), 125–180

[80] B. Keller and I. Reiten, Cluster-tilted algebras are Gorenstein and stably Calabi-Yau. *Adv. Math.* **211** (2007), 123–151

[81] B. Keller and D. Yang, Derived equivalences from mutations of quivers with potential. *Adv. Math.* **226** (2011), 2118–2168

[82] Y. Kimura and F. Qin, Graded quiver varieties, quantum cluster algebras and dual canonical basis. *Adv. Math.* **262** (2014), 261–312

[83] H. Krause, Localization theory for triangulated categories. In *Triangulated categories*, pp. 161–235, London Math. Soc. Lecture Note Ser. 375, Cambridge Univ. Press, Cambridge, 2010, URL https://doi.org/10.1017/CBO9781139107075.005

[84] M. C. Kulkarni, Dimer models on cylinders over Dynkin diagrams and cluster algebras. *Proc. Amer. Math. Soc.* **147** (2019), 921–932

[85] D. Labardini-Fragoso, Quivers with potentials associated to triangulated surfaces. *Proc. Lond. Math. Soc. (3)* **98** (2009), 797–839

[86] I. Le, Cluster structures on higher Teichmuller spaces for classical groups. *Forum Math. Sigma* **7** (2019), Paper No. e13, 165

[87] B. Leclerc, Cluster structures on strata of flag varieties. *Adv. Math.* **300** (2016), 190–228

[88] K. Lee and R. Schiffler, Positivity for cluster algebras. *Annals of Mathematics* (2015), 73–125

[89] K. Lee and R. Schiffler, Cluster algebras and Jones polynomials. *Selecta Math. (N.S.)* **25** (2019), Paper No. 58, 41

[90] T. Magee, Littlewood-Richardson coefficients via mirror symmetry for cluster varieties. *Proc. Lond. Math. Soc. (3)* **121** (2020), 463–512

[91] T. Mandel, Theta bases and log Gromov-Witten invariants of cluster varieties. *Trans. Amer. Math. Soc.* **374** (2021), 5433–5471

[92] B. R. Marsh and Y. Palu, Coloured quivers for rigid objects and partial triangulations: the unpunctured case. *Proc. Lond. Math. Soc. (3)* **108** (2014), 411–440

[93] S. Morier-Genoud, V. Ovsienko and S. Tabachnikov, 2-frieze patterns and the cluster structure of the space of polygons. *Ann. Inst. Fourier (Grenoble)* **62** (2012), 937–987

[94] G. Musiker and R. Schiffler, Cluster algebras of unpunctured surfaces and snake graphs. In *21st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2009)*, pp. 673–684, Discrete Math. Theor. Comput. Sci. Proc. AK, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2009

[95] G. Musiker and R. Schiffler, Cluster expansion formulas and perfect matchings. *J. Algebraic Combin.* **32** (2010), 187–209

[96] G. Musiker, R. Schiffler and L. Williams, Positivity for cluster algebras from surfaces. *Adv. Math.* **227** (2011), 2241–2308

[97] G. Musiker, R. Schiffler and L. Williams, Bases for cluster algebras from surfaces. *Compos. Math.* **149** (2013), 217–263

[98] W. Nagai and Y. Terashima, Cluster variables, ancestral triangles and Alexander polynomials. *Adv. Math.* **363** (2020), 106965, 37

[99] K. Nagao, Donaldson-Thomas theory and cluster algebras. *Duke Math. J.* **162** (2013), 1313–1367

[100] M. Pressland, Mutation of frozen Jacobian algebras. *J. Algebra* **546** (2020), 236–273

[101] J. Propp, The combinatorics of frieze patterns and Markoff numbers. *Integers* **20** (2020), Paper No. A12, 38

[102] Y. Qiu and Y. Zhou, Cluster categories for marked surfaces: punctured case. *Compos. Math.* **153** (2017), 1779–1819

[103] N. Reading, Universal geometric cluster algebras from surfaces. *Trans. Amer. Math. Soc.* **366** (2014), 6647–6685

[104] I. Reiten, Cluster categories. In *Proceedings of the International Congress of Mathematicians. Volume I*, pp. 558–594, Hindustan Book Agency, New Delhi, 2010

[105] K. Rietsch and L. Williams, Newton-Okounkov bodies, cluster duality, and mirror symmetry for Grassmannians. *Duke Math. J.* **168** (2019), 3437–3527

[106] V. Schechtman, Pentagramma mirificum and elliptic functions (Napier, Gauss, Poncelet, Jacobi, ...). *Ann. Fac. Sci. Toulouse Math. (6)* **22** (2013), 353–375

[107] R. Schiffler, A geometric model for cluster categories of type D_n. *J. Algebraic Combin.* **27** (2008), 1–21

[108] R. Schiffler, Cluster algebras from surfaces. In *Homological methods, representation theory, and cluster algebras*, pp. 65–99, Springer, 2018

[109] R. Schiffler and H. Thomas, On cluster algebras arising from unpunctured surfaces. *Int. Math. Res. Not. IMRN* (2009), 3160–3189

[110] J. Scott, Grassmannians and cluster algebras. *Proc. London Math. Soc. (3)* **92** (2006), 345 – 380

[111] V. Shende, D. Treumann, H. Williams and E. Zaslow, Cluster varieties from Legendrian knots. *Duke Math. J.* **168** (2019), 2801–2871

[112] E. Tzanaki, Polygon dissections and some generalizations of cluster complexes. *J. Combin. Theory Ser. A* **113** (2006), 1189–1198

[113] H. Williams, Cluster ensembles and Kac-Moody groups. *Adv. Math.* **247** (2013), 1–40

[114] B. Zhu, BGP-reflection functors and cluster combinatorics. *J. Pure Appl. Algebra* **209** (2007), 497–506