The VUB-Leeds lectures on Algebraic and categorical structures for modular functors and quantum codes

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The current version of these notes can be found under
`https://christophschweigert.github.io/2026.bruessel.pdf`
as a pdf file.

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1 Lecture 1: Semisimple Hopf algebras and the Kitaev model

In these lectures, K is an algebraically closed field of characteristic 0, why not $\mathbb{C}$. Vector spaces will be typically finite-dimensional. Algebras and morphisms of algebras are unital.

1.1 Mathematical background

1.1.1 Hopf algebras and tensor categories

We start to explain some basic algebraic notions from the theory of Hopf algebras. References include [Kas95, S95] and, e.g., the lecture notes [S25].

The category of K-vector spaces is an abelian category. Any K-algebra A gives an abelian category: $A-$mod. The category vect_K of K-vector spaces is even a monoidal category.

Definition 1.1.1

1. *Let $\mathcal{C}$ be a category and $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ a functor, called a tensor product.*

2. *A monoidal category or tensor category consists of a category $(\mathcal{C}, \otimes)$ with tensor product, an object $\mathbb{I} \in \mathcal{C}$, called the tensor unit, and a natural isomorphism, called the associator,*

$$a : \otimes(\otimes \times \ \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes)\,.$$

   of functors $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and natural isomorphisms

$$r : \mathrm{id} \otimes \mathbb{I} \to \mathrm{id} \quad \text{and} \quad l : \mathbb{I} \otimes \mathrm{id} \to \mathrm{id}$$

   called unit constraints such that the following axioms hold:

   - *The pentagon axiom: for all quadruples of objects $U, V, W, X \in \mathrm{Obj}(\mathcal{C})$ the following diagram commutes*

$$\begin{array}{ccccc} & & (U\otimes V)\otimes(W\otimes X) & & \\ & \nearrow^{a_{U\otimes V,W,X}} & & \searrow^{a_{U,V,W\otimes X}} & \\ ((U\otimes V)\otimes W)\otimes X & & & & U\otimes(V\otimes(W\otimes X)) \\ \downarrow{\scriptstyle a_{U,V,W}\otimes \mathrm{id}_X} & & & & {\scriptstyle \mathrm{id}_U\otimes a_{V,W,X}}\uparrow \\ (U\otimes(V\otimes W))\otimes X & & \xrightarrow{\quad a_{U,V\otimes W,X}\quad} & & U\otimes((V\otimes W)\otimes X)\end{array}$$

   - *The triangle axiom: for all pairs of objects $V, W \in \mathrm{Obj}(\mathcal{C})$ the following diagram commutes*

$$\begin{array}{ccc} (V\otimes\mathbb{I})\otimes W & \xrightarrow{\quad a_{V,\mathbb{I},W}\quad} & V\otimes(\mathbb{I}\otimes W) \\ & {\scriptstyle r_V\otimes \mathrm{id}_W}\searrow \quad \swarrow{\scriptstyle \mathrm{id}_V\otimes l_W} & \\ & V\otimes W & \end{array}$$

For an introduction to category theory, we refer to [Riehl] and for monoidal categories to [Kas95, TV17, EGNO15]. For an elementary introduction to category-theoretic notions in the context of topological field theories, we refer to [FSW26].

Observation 1.1.2. • Other examples of monoidal categories are the category of representations of a group or of a Lie algebra.

- Which structure can turn $A-$mod into a monoidal category? Given an algebra morphism $\Delta : A \to A \otimes A$, the pullback functor

$$\Delta^* : A\text{-mod} \boxtimes A\text{-mod} \cong (A \otimes A)\text{-mod} \to A\text{-mod}$$

together with the associators of vect_K endows $A-$mod with the structure of a monoidal category.

- For groups, we have a one-dimensional representation which is essential for defining (co)invariants and thus for group cohomology. We therefore require the existence of an algebra morphism

$$\epsilon : A \to K \ .$$

A representation with $a.v = \epsilon(a)v$ is said to be invariant under A. The corresponding one-dimensional representation forms a monoidal unit for $A-$mod, iff

$$(\epsilon \otimes \mathrm{id}_A) \circ \Delta = \mathrm{id}_A = (\mathrm{id}_A \otimes \epsilon) \circ \Delta \tag{1}$$

We have arrived at the notion of a bialgebra.

Definition 1.1.3

1. *A triple (A, μ, Δ) is called a* *bialgebra*, *if*
   - *(A, μ) is an associative algebra, having a unit $\eta : K \to A$.*
   - *(A, Δ) is a coassociative coalgebra, having a counit $\epsilon : A \to K$.*
   - *The coproduct $\Delta : A \to A \otimes A$ is a map of unital algebras.*
   - *The counit $\epsilon : A \to K$ is a map of unital algebras.*
2. *A K-linear map is said to be a* *bialgebra map*, *if it is both an algebra and a coalgebra map.*

Remarks 1.1.4. 1. The group algebra $K[G]$ of a group G, the tensor algebra $T^\bullet V$ of a vector space V and the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ are bialgebras. In particular, the symmetric algebra $S^\bullet V$ of a vector space is a Hopf algebra.

For later use, we exhibit the group algebra $K[G]$ of a group G as a bialgebra. It is defined on the vector space freely generated by the set underlying a group G and thus has a distinguished basis $\{b_g\}_{g \in G}$ labeled by group elements, in which we have

$$b_g \cdot b_h = b_{gh} \quad \Delta(b_g) = b_g \otimes b_g \quad \epsilon(b_g) = 1$$

and unit b_e. (Sometimes, when the context is unambiguous, we will write g instead of b_g.)

Exercise: Show that the alternating algebra $\mathrm{Alt}^\bullet(V)$ of a finite-dimensional vector space V is a bialgebra in the symmetric monoidal category of super vector spaces.

2. The following notation is due to Heyneman and Sweedler and frequently called Sweedler notation: let (C, Δ, ϵ) be a coalgebra. For any $x \in C$, we can find finitely many elements $x'_i \in C$ and $x''_i \in C$ such that
$$\Delta(x) = \sum_i x'_i \otimes x''_i \ .$$
Dropping the summation indices, this is written as
$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \ .$$
It is common to even suppress the sum and write
$$\Delta(x) = x_{(1)} \otimes x_{(2)} \ .$$
In this notation, counitality (1) reads
$$\epsilon(x_{(1)})x_{(2)} = x_{(1)}\epsilon(x_{(2)}) = x \quad \text{for all } x \in C,$$
and cocommutativity $\Delta(x) = \tau \circ \Delta(x)$ for all $x \in H$ with $\tau : H \otimes H \to H \otimes H$ the map exchanging the tensorands reads
$$x_{(1)} \otimes x_{(2)} = x_{(2)} \otimes x_{(1)} \quad \text{for all } x \in C \ .$$
Finally, coassociativity reads
$$(x_{(1)})_{(1)} \otimes (x_{(1)})_{(2)} \otimes x_{(2)} = x_{(1)} \otimes (x_{(2)})_{(1)} \otimes (x_{(2)})_{(2)} \ .$$
For the sake of a compact notation, we denote this element also by $x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$. We use a similar notation for multiple coproducts.

1.1.2 Rigidity

Finite-dimensional vector spaces have duals with particularly nice properties, e.g. $W \otimes V^* \cong \mathrm{Hom}_K(V, W)$. Their tensor product is exact (not true for infinite-dimensional vector spaces).

Definition 1.1.5

1. *Let $\mathcal{C}$ be a tensor category. An object V of $\mathcal{C}$ is called right dualizable, if there exists an object $V^\vee \in \mathcal{C}$ and morphisms*
$$b_V : \mathbb{I} \to V \otimes V^\vee \quad \textit{and} \quad d_V : V^\vee \otimes V \to \mathbb{I}$$
such that
$$\begin{aligned} r_V \circ (\mathrm{id}_V \otimes d_V) \circ a_{V,V^\vee,V} \circ (b_V \otimes \mathrm{id}_V) \circ l_V^{-1} &= \mathrm{id}_V \\ l_{V^\vee} \circ (d_V \otimes \mathrm{id}_{V^\vee}) \circ a^{-1}_{V^\vee,V,V^\vee} \circ (\mathrm{id}_{V^\vee} \otimes b_V) \circ r^{-1}_{V^\vee} &= \mathrm{id}_{V^\vee} \end{aligned}$$
Such an object $V^\vee$ is called a right dual to V.

The morphism d_V is called an evaluation, the morphism b_V a coevaluation.

2. *A monoidal category is called right-rigid or right-autonomous, if every object has a right dual.*

3. *A left dual to V is an object ${}^\vee V$ of $\mathcal{C}$, together with two morphisms*

$$\tilde{b}_V : \mathbb{I} \to {}^\vee V \otimes V \quad \text{and} \quad \tilde{d}_V : V \otimes {}^\vee V \to \mathbb{I}$$

such that analogous equations hold. A left-rigid or left autonomous category is a monoidal category in which every object has a left dual.

4. *A monoidal category is rigid or autonomous, if it is both left and right rigid.*

Remarks 1.1.6. 1. A K-vector space V has a right dual, if and only if it is finite-dimensional.

2. If H is a finite-dimensional Hopf algebra, then its dual vector space H^* has a natural structure of a finite-dimensional Hopf algebra as well. For example, multiplication of $\alpha, \beta \in H^*$ is defined as the dual map to comultiplication in H, i.e.

$$(\alpha \cdot \beta)(h) = (\alpha \otimes \beta) \circ \Delta(h)$$

For the group algebra $K[G]$ with G a finite group, we set $K(G) := K[G]^*$ with the dual basis δ_g with $\delta_g(b_h) = \delta_{g,h}$. One computes

$$\delta_{g_1}\delta_{g_2}(b_h) = (\delta_{g_1} \otimes \delta_{g_2})(\Delta(b_h)) = \delta_{g_1}(h)\delta_{g_2}(h) = \delta_{g_1,g_2}\delta_{g_1}(h)$$

and thus

$$\delta_{g_1}\delta_{g_2} = \delta_{g_1,g_2}\delta_{g_1} \quad \Delta(\delta_g) = \sum_{g_1,g_2,g_1g_2=g} \delta_{g_1} \otimes \delta_{g_2} \quad \epsilon(\delta_h) = \delta_h(1) = \delta_{h,e} \ .$$

The unit element is $1_{H^*} = \sum_{h \in H} \delta_h$.

3. Let $\mathcal{C}$ be an abelian monoidal category. Suppose that the object X is rigid. Then the functors $- \otimes X$ and $X \otimes -$ are exact.

Definition 1.1.7

1. *We say that a bialgebra (H, μ, Δ) is a Hopf algebra, if the identity id_H has a two-sided inverse $S : H \to H$ under the convolution product:*

$$x_{(1)} \cdot S(x_{(2)}) = \epsilon(x) \cdot 1 = S(x_{(1)}) \cdot x_{(2)} \ .$$

This inverse is then called the antipode of the Hopf algebra. If an antipode exists, it is, as a two-sided inverse for an associative product, uniquely determined.

2. *An inverse $\overline{S} : H \to H$ under the twisted convolution product*

$$x_{(2)} \cdot \overline{S}(x_{(1)}) = \epsilon(x) \cdot 1 = \overline{S}(x_{(2)}) \cdot x_{(1)}$$

is called a skew antipode.

Remarks 1.1.8. 1. By a theorem of Larson and Sweedler, the antipode of a finite-dimensional Hopf algebra is an invertible linear map. Its inverse is a skew antipode.

2. The category $H-$mod of finite-dimensional modules over a Hopf algebra is right rigid, due to the antipode. If a skew antipode exists, it is left rigid.

3. If G is a group, the group algebra $K[G]$ is a Hopf algebra with antipode

$$S(g) = g^{-1} \quad \text{for all} \quad g \in G \ .$$

Indeed, we have for $g \in G$:

$$\mu \circ (S \otimes \mathrm{id}) \circ \Delta(g) = \mu \circ (S \otimes \mathrm{id})(g \otimes g) = g^{-1} \cdot g = \epsilon(g)1 = 1 \ .$$

(Here, we replaced the notation for the basis element b_g by g.) In this case, we have a cocommutative Hopf algebra and antipode and skew antipode coincide.

4. In the same way, taking the inverse in a group reverses the order of multiplication, $(gh)^{-1} = h^{-1}g^{-1}$, the antipode and the skew antipode can be shown to be morphisms $A \to A^{opp}$.

1.1.3 Integrals and Frobenius algebras

The following notion will be very important for us:

Definition 1.1.9

1. *Let H be a Hopf algebra. The K-linear subspace*

$$\mathcal{I}_l(H) := \{x \in H \,|\, h \cdot x = \epsilon(h)x \quad \text{for all } h \in H\}$$

is called the space of <u>left integrals</u> of the Hopf algebra H. Similarly,

$$\mathcal{I}_r(H) := \{x \in H \,|\, x \cdot h = \epsilon(h)x \quad \text{for all } h \in H\}$$

is called the space of <u>right integrals</u> of H.

2. *Similarly, the subspace of the linear dual H^**

$$C\mathcal{I}_l(H) := \{\phi \in H^* \,|\, (\mathrm{id}_H \otimes \phi) \circ \Delta_H(h) = 1_H\phi(h) \quad \text{for all } h \in H\}$$

is called the space of <u>left cointegrals</u>. Right cointegrals are defined analogously.

3. *A Hopf algebra is called <u>unimodular</u>, if $\mathcal{I}_l(H) = \mathcal{I}_r(H)$.*

Remarks 1.1.10.

1. Consider the canonical isomorphism $\mathrm{Hom}_K(K, H) \cong H$. Under this identification, an integral $\Lambda \in \mathcal{I}_l(H) \subset H$ corresponds to a morphism of left H-modules in $\mathrm{Hom}_H(K, H) \subset \mathrm{Hom}_K(K, H)$ from the trivial H-module to the left regular H-module. A similar statement holds for right integrals.

2. Similarly, a left cointegral is a morphism $\phi : H \to K$ which is a morphism of left comodules. Here, the coproduct endows H with the structure of a regular left H-comodule while K become a left comomodule via $K \to H \otimes K$ with $\lambda \mapsto 1_H \otimes \lambda$.

3. Even if a Hopf algebra H is cocommutative, it can be not unimodular. For an example, see [M93, p. 17].

4. Let G be a *finite* group. Then the group algebra $K[G]$ is a unimodular Hopf algebra, with integral
$$\mathcal{I}_l = \mathcal{I}_r = K \sum_{g \in G} g \ .$$
Indeed, for $I := \sum_{h \in G} h$ we have $g.I = \sum_{h \in G} gh = I = \epsilon(g) I$ for all $g \in G$, and it is enough to check this relation on the distinguished basis of $K[G]$.

5. Similarly, the cointegral of $K[G]$ is $\lambda = \delta_e \in H^*$, where $\delta_e(h) = \delta_{e,h}$.

6. Exercise: show that the (left and right) cointegral of the alternating algebra $\mathrm{Alt}^\bullet(V)$ of a finite-dimensional vector space V, seen as a Hopf algebra in super vector spaces, is given by top degree elements.

7. Let M be an H-module and $m \in \mathrm{Im}\,(\Lambda) \subset M$, i.e. $m = \Lambda.m'$. Then for any $h \in H$, we have
$$h.m = h.\Lambda m' = \epsilon(h)\Lambda m' = \epsilon(h) m \ ,$$
i.e. m is invariant under the action of H.

Theorem 1.1.11 (Maschke)**.**
Let H be a finite-dimensional Hopf algebra. Then the following statements are equivalent:

1. H is semisimple.

2. The counit takes non-zero values on the one-dimensional space of left integrals, $\epsilon(\mathcal{I}_l(H)) \neq 0$.

Remarks 1.1.12. 1. Let H be semisimple. Then there is a unique element $\Lambda \in \mathcal{I}_l(H)$ such that $\epsilon(\Lambda) = 1$, called the <u>Haar integral</u>.

2. A Haar integral is an idempotent in H:
$$\Lambda \cdot \Lambda = \epsilon(\Lambda)\Lambda = \Lambda$$

3. For a left integral $\Lambda \in \mathcal{I}_l(H)$, also Λh is a left integral for any $h \in H$. Since the space of integrals is one-dimensional, we can define a linear form $\alpha \in H^*$, called the distinguished group-like element of H^*, by
$$\Lambda h = \alpha(h)\Lambda \quad \text{for all } h \in H$$
Unimodularity is equivalent to $\alpha = \epsilon$.

4. A finite-dimensional semisimple Hopf algebra is unimodular.

   Suppose that H is semisimple and Λ a Haar integral, i.e. $\epsilon(\Lambda) = 1$. Then for any $h \in H$, we have
$$\alpha(h)\epsilon(\Lambda)\Lambda = \alpha(h)\Lambda^2 = (\Lambda h)\Lambda = \Lambda(h\Lambda) = \epsilon(h)\Lambda^2 = \epsilon(h)\epsilon(\Lambda)\Lambda \ ,$$
where we used the definition of a left integral and of the distinguished group-like element α of H^*. Since $\epsilon(\Lambda) \neq 0$, we have $\alpha(h) = \epsilon(h)$ for all $h \in H$ which implies unimodularity.

We need a famous theorem:

Theorem 1.1.13 (Larson-Radford, 1988)**.**
Let K be a field of characteristic zero. Let H be a finite-dimensional K-Hopf algebra. Then the following statements are equivalent:

1. H is semisimple.
2. H^* is semisimple.
3. $S^2 = \mathrm{id}_H$.

We need another notion:

Definition 1.1.14

1. *Given an associative unital algebra (A, μ, η) in a monoidal category $\mathcal{C}$, the structure of a Frobenius algebra is a counital coproduct $\Delta_F : A \to A \otimes A$ and a counit $\epsilon_F : A \to 1_{\mathcal{C}}$ which are morphisms of bimodules. (Equivalently, one can require the existence of a Frobenius form $\epsilon_F : A \to 1_F$ such that the bilinear form $\kappa_F = \epsilon_F \circ \mu$ is non-degenerate.)*
2. *A Frobenius algebra in the category of vector spaces is called symmetric, if the non-degenerate bilinear form κ_F is symmetric. (Symmetric Frobenius algebras can be defined more generally in pivotal tensor categories.)*

Remarks 1.1.15. 1. The cointegral allows to endow a finite-dimensional Hopf algebra H with the structure of a Frobenius algebra; similarly for H^* via the integral of H:

$$\kappa(\alpha, \beta) = \alpha \cdot \beta(\Lambda) = \alpha(\Lambda_{(1)}) \cdot \beta(\Lambda_{(2)})$$

For a Frobenius algebra (H, κ), the Nakayama automorphism $\rho : H \to H$ is defined by

$$\kappa(a, b) = \kappa(\rho(b), a) \quad \text{for all } a, b \in H \ .$$

A Frobenius algebra is symmetric, iff ρ is the identity. For Hopf algebras, there is a formula [S95, Proposition 3.6], [S25, Lemma 3.3.9.2]

$$\rho(h) = \alpha(h_{(1)}) S^{-2}(h_{(2)})$$

which for semisimple Hopf algebras by unimodularity $\alpha = 1$ and the Larson-Radford theorem 1.1.13 evaluates to

$$\rho(h) = \epsilon(h_{(1)}) h_{(2)} = h$$

Thus a semisimple Hopf algebra has a canonical structure of a symmetric Frobenius algebra. (For a more general statement in the context of tensor categories, see [FS10, Appendix A2].)

2. We now compute for $\alpha, \beta \in H^*$

$$(\alpha \otimes \beta)\Delta(\Lambda) = \alpha\beta(\Lambda) = \kappa(\alpha, \beta) = \kappa(\beta, \alpha) = \beta\alpha(\Lambda) = (\alpha \otimes \beta)\Delta^{op}(\Lambda)$$

so that integrals of semisimple Hopf algebras are cocommutative, $\Delta(\Lambda) = \Delta^{opp}(\Lambda)$.

1.1.4 Cobordisms

Definition 1.1.16 *Let d be any positive integer. The category $\mathrm{Cob}_{d,d-1}$ of d-dimensional oriented cobordisms (synonymously also called bordisms) is as follows:*

1. *An object of $\mathrm{Cob}_{d,d-1}$ is a closed oriented smooth manifold of dimension $d{-}1$.*
2. *Given a pair of objects $M, N \in \mathrm{Cob}_{d,d-1}$, a morphism $M \to N$ is an equivalence class, with respect to the equivalence relation specified in (3) below, of cobordisms from M to N.*

   Or, spelled out in more detail:

   - *Let M and N be closed oriented $(d{-}1)$-dimensional smooth manifolds. A d-dimensional* cobordism *from M to N is an oriented, d-dimensional smooth manifold B with boundary, together with an orientation preserving diffeomorphism*

     $$\phi_B\colon \quad \overline{M} \sqcup N \xrightarrow{\simeq} \partial B \,. \tag{2}$$

     Here $\overline{M}$ is the same manifold as M but with opposite orientation.
   - *Two cobordisms B and B' from M to N are defined to be equivalent, and thus to represent the same morphism in $\mathrm{Cob}_{d,d-1}$, if there is an orientation-preserving diffeomorphism $\phi\colon B \to B'$ such that the diagram*

     $$\begin{array}{ccc} B & \xrightarrow{\quad\phi\quad} & B' \\ & {\scriptstyle\phi_B}\nwarrow \quad\quad \nearrow{\scriptstyle\phi'_B} & \\ & \overline{M} \sqcup N & \end{array} \tag{3}$$

     commutes.
3. *The identity morphism for an object $M \in \mathrm{Cob}_{d,d-1}$ is represented by the cylinder $B = M \times [0,1]$.*
4. *Composition of morphisms in $\mathrm{Cob}_{d,d-1}$ is given by* gluing *cobordisms: Given three objects $M, M', M'' \in \mathrm{Cob}_{d,d-1}$ and two cobordisms $B\colon M \to M'$ and $B'\colon M' \to M''$, the composition is defined to be the morphism that is represented by the manifold $B \sqcup_{M'} B'$.*

   (To get a smooth structure on this manifold, the gluing must actually be performed along a cylinder over M', often called a collar. *Different choices of collars lead to different glued cobordisms. However, these are all diffeomorphic. We refer to Section 1.3 of [K03] for the technical details.)*
5. *Disjoint union endows the category $\mathrm{Cob}_{d,d-1}$ with a monoidal structure; it is actually in a natural way a symmetric monoidal category.*

Definition 1.1.17 *Let K be an algebraically closed field. A topological field theory of dimension d is a symmetric monoidal functor*

$$\mathrm{tft}\colon : \quad \mathrm{Cob}_{d,d-1} \to \mathrm{vect}(K)\,. \tag{4}$$

For the proof of the following theorem, we refer to the book [K03]:

Theorem 1.1.18. The groupoid of topological field theories

$$\text{tft} : \text{Cob}_{2,1} \to \text{vect} \ .$$

is equivalent to the groupoid of K-Frobenius algebras.

1.1.5 Some modules over Hopf algebras

We need some distinguished modules of a Hopf algebra H.

Like any associative algebra, H is a bimodule over itself. We use the antipode which is an algebra-antimorphism to get two commuting left H-actions

$$L_+(h).x := ax \quad \text{and} \quad L_-(h).x := xS(a) \quad \text{with } h, x \in H \ .$$

In the case of a group algebra $K[G]$, we have

$$L_+(b_g)b_h = b_{gh} \quad \text{and} \quad L_-(b_g)b_h = b_{hg^{-1}} \ .$$

Definition 1.1.19 *Like any associative algebra, a Hopf algebra H also acts on its dual vector space H^* by*

$$(h \rightharpoonup h^*)(k) := h^*(kh) \quad \text{for} \quad h, k \in H, h^* \in H^* \ .$$

The dual H^* is also an algebra, via the coproduct, and we get an action on the bidual.

$$(h^* \rightharpoonup h^{**})(\beta) = h^{**}(\beta \cdot h^*) = \beta h^*(h) = \beta(h_{(1)})h^*(h_{(2)})$$

For finite-dimensional H, we can identify H and the bidual H^{**} which suggests the definition:

Definition 1.1.20

1. *For any Hopf algebra H^* has a natural structure of an algebra. The Hopf algebra H is a left module over H^* via*
$$h^* \rightharpoonup h = h^*(h_{(2)})h_{(1)} \ .$$
2. *We change the notation and have two commuting left H^*-actions on H:*
$$T_+(\beta).x := \langle \beta, x_{(2)} \rangle x_{(1)} \quad \text{and } T_-(\beta).x := \langle \beta, S(x_{(1)}) \rangle x_{(2)} \quad \text{with } x \in H, \beta \in H^* \ .$$

In the case of a group algebra, we have

$$T_+(\delta_g).b_h := \delta_{g,h} b_h \quad \text{and} \quad T_-(\beta).x := \delta_{g,h^{-1}} b_h \ .$$

1.1.6 Drinfeld doubles and braidings

We need the following Hopf algebra built from a Hopf algebra H.

Definition 1.1.21
Let H be a finite-dimensional Hopf algebra. Consider the vector space $D(H) := H^* \otimes H$.

- *It gets the structure of a counital coalgebra using the tensor product structure, i.e. for $f \in H^*$ and $a \in H$, we set*

$$\begin{aligned} \epsilon(f \otimes a) &:= \epsilon(a) f(1) \\ \Delta(f \otimes a) &:= (f_{(1)} \otimes a_{(1)}) \otimes (f_{(2)} \otimes a_{(2)}) \in D(H) \otimes D(H) \ . \end{aligned}$$

- *We define an associative multiplication for $a, b \in H$ and $f, g \in H^*$ by*

$$(f \otimes a) \cdot (g \otimes b) := f \cdot \big(g(S^{-1}(a_{(3)})?a_{(1)})\big) \otimes a_{(2)} b \ .$$

  The unit for this multiplication is $\epsilon \otimes 1 \in H^* \otimes H$.

The following results can be established, see e.g. [Kas95, Chapter IX]:

Proposition 1.1.22. 1. For every finite-dimensional Hopf algebra H, this defines a finite-dimensional Hopf algebra $D(H)$, called the Drinfeld double of H, with antipode given in [Kas95, Theorem IX.2.3].

2. The Drinfeld double $D(H)$ is always unimodular; for H unimodular, its integral is the tensor product of the integrals of H and H^*.

3. The Drinfeld center of a semisimple Hopf algebra is again semisimple. (This is only true in characteristic zero, consider $K(G)$in a characteristic that divides the group order.)

Remarks 1.1.23. 1. A $D(H)$-module can be described as a vector space M endowed with an H action $\rho : H \otimes M \to M$ and an H^* action or, equivalently, an H-coaction $\delta : M \to M \otimes H$, obeying a certain compatibility condition. One can show that H itself with the coregular coaction given by the comultiplication and the adjoint H-action

$$h.x := h_{(1)} \cdot x \cdot S(x_{(2)})$$

provides such an $D(H)$-module.

2. If (e_i) is any basis of H with dual basis (e^i) of H^*, then the element

$$R := \sum_i (1_{H^*} \otimes e_i) \otimes (e^i \otimes 1_H) \in D(H) \otimes D(H) \ ,$$

which, by a standard argument, is independent of the choice of basis, is a universal R-matrix for $D(H)$. This means that the $D(H)$-morphisms

$$\begin{aligned} V \otimes W &\to W \otimes V \\ v \otimes w &\mapsto R_2.w \otimes R_1.v \end{aligned}$$

where we use a "Sweedler-like" notation $R = R_1 \otimes R_2$, endow the category $D(H)$-mod with the structure of a braided monoidal category.

1.2 Kitaev model

We now present a toy model for a system providing a quantum code which generalizes Kitaev's toric code [Ki03] (which is literally not suitable for quantum computing since it does not allow for universal gates for which one needs more complicated, nonabelian representations of the braid group). References for a treatment of the Kitaev model are [BMCA13, BK12, YCC22].

1.2.1 Setting up the model

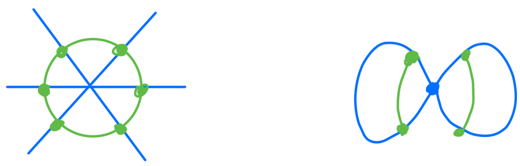
It models (quasi)two-dimensional storage devices. We thus take a compact oriented surface Σ on which the physical degrees of freedom are located on a finite discrete structure:

Definition 1.2.1 *A two-dimensional CW complex Δ is a topological space built by successively attaching 0-dimensional points (vertices), 1-dimensional lines (edges), and 2-dimensional surfaces (faces or plaquettes).*

We assume that

- *The complex Δ is finite.*
- *The CW complex Δ is an oriented topological surface.*
- *The link of every vertex is a circle (i.e., a 1-sphere).*

The link of a vertex $v \in V(X)$ is the graph $Lk(v, \Delta)$ constructed as follows. The vertices of $Lk(v, \Delta)$ are the edges of Δ incident to v. Two such edges are adjacent in $Lk(v, X)$ iff they are incident to a common 2-cell at v.



This excludes for example the bouquet of two circles where the link is not connected.

Then Δ determines a piecewise linear oriented surface Σ. A triangulation of Σ leads to a special example of a 2-dimensional finite CW-complex; the reader does not loose any intuition by keeping this special case in mind.

Our algebraic input datum is a finite-dimensional semisimple Hopf algebra H. We assign degrees of freedom to *oriented* edges e of Δ and consider the K-vector space

$$V(\Sigma, \Delta) := \otimes_{e \in \Delta} H_e \ .$$

Remarks 1.2.2.
1. It is clear that this finite-dimensional vector space depends on the choice of the two-dimensional CW complex Δ.
2. If Δ and Δ' differ by the orientation of an edge e, we use the the antipode on H_e to get an isomorphism
$$V(\Sigma, \Delta) \to V(\Sigma, \Delta')$$
(Note that the antipode squares to the identity.)
3. While the dependence on the additional discrete structure Δ looks like a flaw from the point of view of topological field theories, it is crucial for the application to quantum computing.

For the next definition, we assign

- a copy H_v of H, as a Hopf algebra, to each vertex v of Δ.
- a copy H_f^* of H^*, as a Hopf algebra, to each face f of Δ.

To construct interesting subspaces of $V(\Sigma, \Delta)$, we need linear endomorphisms on V and we obtain them via actions of these two Hopf algebras.

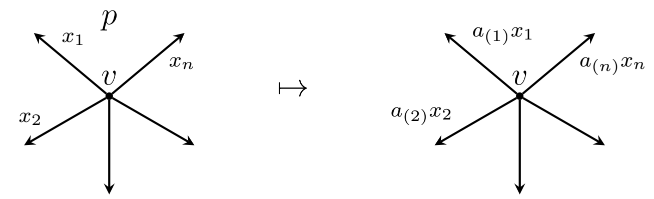
Definition 1.2.3

1. *A* site *of* Δ *is a pair* (v, p)*, consisting of a face* p *and a vertex* v *adjacent to* p*.*
2. *For every site* (v, p) *of* (Σ, Δ) *and every element* $a \in H$*, we define an endomorphism*

$$A_{(v,p)}(a): \quad V(\Sigma, \Delta) \to V(\Sigma, \Delta)$$

by a multiple coproduct and the left action of H *on itself:*

$A_{(v,p)}(a) :=$



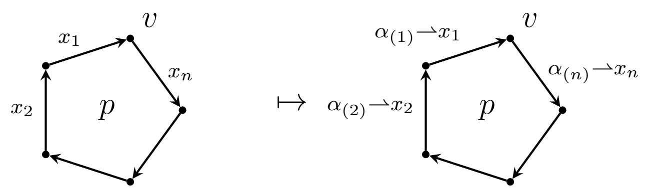
where the edges incident to the vertex v *are indexed counterclockwise starting from the plaquette* p*. Here all edges incident to the vertex* v *are assumed to point away from* v*.*

3. *Given a site* $s = (v, p)$ *of the decomposition* Δ *and every element* $\alpha \in H^*$*, the plaquette operator*

$$B_{(v,p)}(\alpha): \quad V(\Sigma, \Delta) \to V(\Sigma, \Delta)$$

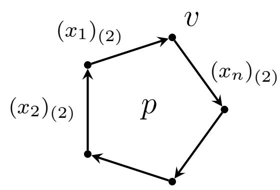
is defined by a multiple coproduct in H^* *and a left action of* H^* *on* H*, where* $\alpha.x = \alpha \rightharpoonup h$*.*

$B_{(v,p)}(\alpha)$:



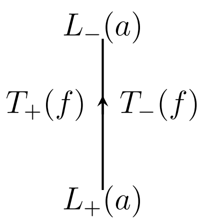
(5)

$$= \langle \alpha, S(x_n)_{(1)} \dots (x_1)_{(1)} \rangle$$



(6)

Schematically, we have:



Remark 1.2.4. We can use the antipode to change the orientation: instead of $L_+(a_{(i)}).x = a_{(i)}x_i$, we have

$$S\left(a_{(i)}S(x_i)\right) = x_iS(a_{(i)}) = L_-(a_{(i)}).x \ .$$

Proposition 1.2.5.

1. For every site (v, p), the assignment $a \mapsto A_{p,v}(a)$ defines an action $\rho_{v,p}$ of H_v on $V(\Sigma, \Delta)$.

2. For every site (v, p), the assignment $\alpha \mapsto A_{p,v}(\alpha)$ defines an action $\tilde{\rho}_{v,p}$ of H_p^* on $V(\Sigma, \Delta)$.

3. If v, w are distinct vertices of Δ, then the H-actions $\rho_{v,p}$ and $\rho_{w,p'}$ commute.

4. Similarly, if p, q are distinct plaquettes, then the H^* actions $\tilde{\rho}_{v,p}$ and $\tilde{\rho}_{v',q}$ commute.

5. If the sites are different, then the operators $A_{(v,p)}(h)$ and $B_{(v',p')}(\alpha)$ commute.

Proof.

1. The operators $A^-_{(v,-)}$, $A^-_{(w,-)}$ obviously commute if the edges incident to the vertex v and those incident to the vertex w are disjoint. We therefore assume that the vertices v and w are adjacent, i.e. at least one edge connects them. Clearly, we need only to check that the actions of $A^-_{(v,-)}$ and $A^-_{(w,-)}$ commute on their common support. Suppose such an edge e is oriented so that it points from the vertex v to the vertex w. Then $A^-_{(v,-)}$ acts on the corresponding copy of H via the left regular representation, and $A^-_{(w,-)}$ acts on the copy of H associated to the edge e via the right regular representation. These are commuting actions by associativity.

2. This statement is dual to 1, using coassociativity.

3. Follows by the same type of argument.

□

We need the following Lemma to understand the situation at a site (p, v)

Lemma 1.2.6.
Let X be any H-module and Y any H^*-module. For $h \in H$, $\alpha \in H^*$, define the endomorphisms $p_h, q_\alpha \in \mathrm{End}(H \otimes X \otimes Y \otimes H)$ by

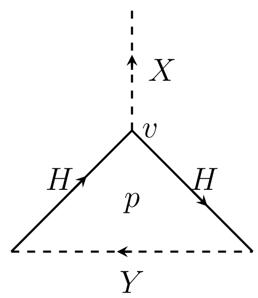
$$p_h(u \otimes x \otimes y \otimes v) = h_{(1)}u \otimes h_{(2)}x \otimes y \otimes vS(h_{(3)})$$
$$q_\alpha(u \otimes x \otimes y \otimes v) = \alpha_{(3)} \rightharpoonup u \otimes x \otimes \alpha_{(2)} \rightharpoonup y \otimes \alpha_{(1)} \rightharpoonup v$$

Then

$$D(H) \to \mathrm{End}(H \otimes X \otimes Y \otimes H)$$
$$h \otimes \alpha \mapsto p_h q_\alpha$$

is a morphism of algebras.

The following drawing illustrates the situation:



The H-module X is a placeholder for all edges starting or ending at the vertex v that are not adjacent to the plaquette p. Similarly, the H^* module Y is a placeholder for edges adjacent to the plaquette p, but not to the vertex v.

Proof.
It is clear that we get via $h \mapsto p_h$ the structure of an H-module, via $\alpha \mapsto q_\alpha$ the structure of an H^*-module. The straightening formula

$$a \cdot f = f(S^{-1}a_{(3)}?a_{(1)}) \cdot a_{(2)} \ .$$

is shown in a detailed calculation in [BMCA13, Lemma 1, Theorem 1]. The reader should also revisit remark 1.1.23.2 before reading the proof.

Instead, we check the statement for the case of group algebras, $H = K[G]$. We find

$$\begin{aligned} p_g q_h(u \otimes x \otimes y \otimes v) &= \textstyle\sum_{h_1 h_2 h_3 = h} p_g(\delta_{h_3} \rightharpoonup u \otimes x \otimes \delta_{h_2} \rightharpoonup y \otimes \delta_{h_1} \rightharpoonup v) \\ &= p_g(u \otimes x \otimes y \otimes v)\delta_{uyv,h} \\ &= gu \otimes gx \otimes y \otimes vg^{-1}\delta_{uyv,h} \end{aligned}$$

On the other hand, we find

$$\begin{aligned} q_{ghg^{-1}} p_g(u \otimes x \otimes y \otimes v) &= q_{ghg^{-1}}(gu \otimes gx \otimes y \otimes vg^{-1}) \\ &= gu \otimes gx \otimes y \otimes vg^{-1}\delta_{guyvg^{-1},ghg^{-1}} \end{aligned}$$

□

We now get:

Theorem 1.2.7.
For a given site $s = (v,p)$, the operators $A_{(v,p)}(h)$ and and $B_{(v,p)}(\alpha)$ satisfy the commutation relations of the Drinfeld double $Z(H)$: the map

$$\begin{aligned} \rho_s \colon D(H) &\to \mathrm{End}(V(\Sigma, \Delta)) \\ a \otimes \alpha &\mapsto A_{(v,p)}(a) B_{(v,p)}(\alpha) \end{aligned}$$

is an algebra morphism.

1.2.2 Definition the Hamiltonian

Observation 1.2.8.

1. Let $h \in H$ be a cocommutative element, i.e. $\Delta(h) = \Delta^{opp}(h)$. Then the multiple coproduct $\Delta^{(n)}(h) \in H^{\otimes n}$ is cyclically invariant. As a consequence, the endomorphism $A_{(s,p)}(h)$ is independent of the plaquette p which was previously used to construct a linear order on the edges incident to the vertex v. We denote the endomorphism by $A_s(h)$. Similarly, $B_p(f)$ for a cocommutative element $f \in H^*$ is independent on the vertex.

2. Both H and H^* have, as semisimple Hopf algebras, Haar integrals which are cocommutative by remark 1.1.15.4. We thus get an endomorphism $A_v := A_v(\Lambda)$ for each vertex and $B_p := B_p(\lambda)$ for each plaquette.

Lemma 1.2.9.
The endomorphisms A_v and B_p are commuting idempotents:

$$A_v^2 = A_v \qquad A_v B_p = B_p A_v \qquad \text{and} \qquad B_p^2 = B_p \ .$$

Proof.
For a normalized integral, we have $\Lambda \cdot \Lambda = \epsilon(\Lambda)\Lambda = \Lambda$. Theorem 1.2.7 now implies that the endomorphisms are idempotents. A two-sided integral is central, $\Lambda \cdot h = \epsilon(h)\Lambda = h \cdot \Lambda$ for all $h \in H$, which implies again with theorem 5.5.16 that the endomorphisms commute. $\square$

The following sum of commuting idempotents on $V(\Sigma, \Delta)$, called the Hamiltonian:

$$\mathsf{H} := \sum_v (\mathrm{id} - A_v) + \sum_p (\mathrm{id} - B_p)$$

is diagonalizable. Idempotents have eigenvalues 0 and 1, so H is diagonalizable with eigenvalues in $\mathbb{Z}_{\geq 0}$. When we interpret H as a Hamiltonian, then these would be the energy spectrum; such Hamiltonians are not realistic.

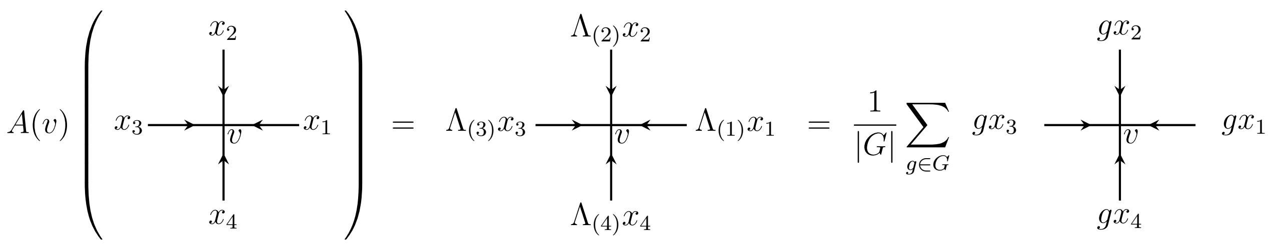
Definition 1.2.10
The ground state or protected subspace is the zero eigenspace of H*:*

$$Z_K(H; \Sigma, \Delta) := \{v \in H^{\otimes n} : \mathsf{H}v = 0\} \subset V(\Sigma, \Delta)$$

Remarks 1.2.11.
1. Using the fact that the idempotents commute, one shows that $x \in Z_K(H; \Sigma, \Delta)$, if and only if $A_v x = x$ and $B_p x = x$ for all vertices v and all plaquettes p.
2. The orientation of the edges does not really matter: if one reverses an edge, the action of the antipode gives an appropriate isomorphism.
3. Using Theorem 3.1.7, we will see that for different two-dimensional CW-complexes for the same PL manifold Σ, we have canonical isomorphisms $Z_K(H; \Sigma, \Delta) \cong Z_K(H; \Sigma, \Delta)$.
4. Exchanging H and H^* and applying Poincaré duality to Δ, we then expect that H and H^* give isomorphic ground states. This can be seen as a first hint that the construction is Morita invariant, where we have a categorified notion of Morita invariance for (pivotal) monoidal categories, see e.g.. [FGJS25].
5. We have arrived at a system, consisting of a subspace $Z_K(H; \Sigma, \Delta) \subset V(\Sigma, \Delta)$. This is a situation we encounter for codes.

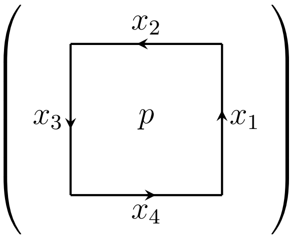
Example 1.2.12. In the case of a group algebra of a finite group, $H = \mathbb{C}[G]$, we use the distinguished basis $(b_g)_{g \in G}$ of H consisting of group elements of G. (Kitaev's toric code uses the cyclic group $\mathbb{Z}_2$.) A basis of $V(\Sigma, \Delta)$ is given by assigning to any edge of Δ a group element $x \in G$. We interpret the group elements g as the holonomy of a G-connection along the edge. (For a finite group G, this means that we are talking about G-principal bundles.)

- By the properties of the integral, a ground state is invariant under the H-action at any vertex:



where $x_i \in G$. The projection by the operator A_v implements gauge invariance at the vertex v by averaging with respect to the Haar integral $\frac{1}{|G|}\sum_g b_g$ of $K[G]$.

- Similarly, we get for

$B(p)$


a mutiple of the state, but with prefactor, cf. equation (5) which is

$$\langle \lambda, S(g_n \cdot g_1) \rangle = \delta_{g_1 \cdots g_n, e}$$

The projection by the operator B_p implements that locally on the face p the holonomy vanishes i.e. that the connection is flat.

- One can thus show that for $K[G]$, the space of ground states on a closed oriented surface Σ is the linear spann of $\mathrm{Hom}_{Grp}(\pi_1(\Sigma), G)/G$, i.e. the space of functions on the so-called character variety. For a finite abelian group A, this is

$$\mathrm{Hom}_{Grp}(\pi_1(\Sigma_g), A) \cong \mathrm{Hom}_{Grp}(H_1(\Sigma_g, \mathbb{Z}), A) \cong \mathrm{Hom}_{Grp}(\mathbb{Z}^{2g}, A) \ .$$

As a consequence, we find in this case

$$\dim_K Z_K(K[A]; \Sigma, \Delta) = |A|^{2g} \ ,$$

which obviously depends on the genus of Σ.

1.2.3 Anyons and excited states

Observation 1.2.13.

- Fix a collection S of finitely many disjoint sites (v_i, p_i). This means that the vertex v_i is only incident to the plaquette p_i and no other plaquette so that the $D(H)$-actions for all plaquettes commute. Write $p \notin S$, if a plaquette is not one of the p_i, and similarly $s \notin S$ for vertices.

- Define an endomorphism

$$\mathsf{H}_S : \quad V(\Sigma, \Delta) \to V(\Sigma, \Delta)$$

by

$$\mathsf{H}_S := \sum_{v \notin S} (\mathrm{id} - A_v) + \sum_{p \notin S} (\mathrm{id} - B_p) \ ;$$

this is again a sum of commuting idempotents. We introduce the auxillary space

$$K(\Delta, S) := \ker \mathsf{H}_S = \{x \in V(\Sigma, \Delta) |\, A_v x = x \text{ for all } v \notin S \text{ and } B_p x = x \text{ for all } p \notin S\}$$

- At each site $(v_i, p_i) \in S$, we have an action of $D(H)$; these actions commute, so $K(\Delta, S)$ is an $D(H)^{\otimes |S|}$-module which we decompose in isotypical components:

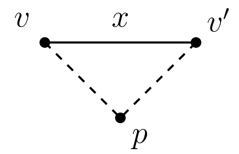
$$K(\Delta, s_1, \ldots s_n) = \oplus_{M_1, \ldots M_n} (M_1^\vee \boxtimes M_2^\vee \ldots M_n^\vee) \otimes Z_K(H; \Sigma, Y_1, \ldots Y_n)$$

The algebra $D(H)^{\otimes |S|}$ acts trivially on the vector space $Z_K(H; \Sigma, Y_1, \ldots Y_n)$. It is called a protected subspace. It can be endowed with the action of a mapping class group of Σ. We will see this vector space in many disguises.

We finally need operators to create anyons and to move them. These are called ribbon operators which depend on $h \in H$ and $\beta \in H^*$.

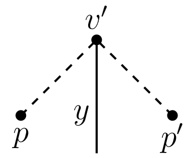
Observation 1.2.14. We want to move an anyon attached to a site (v, p) to a neighboring site. There are two elementary situations :

- We keep the plaquette p, but go to a neighboring vertex v':



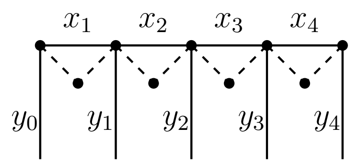
  Dashed lines connecting a vertex and a plaquette indicate the two sites (v, p) and (v', p). The solid line is a 1-cell in Δ. Then we map for $h \in H$ and $\beta \in H^*$ the element $x \in H$ to $F^{h,\beta}(x) = \epsilon(h)f(Sx_{(2)})x_{(1)}$.

- We keep the vertex v, but go to a neighboring plaquette p':



  (It is helpful to keep in mind that p and p' are connected by an edge in the Poincaré dual complex.) Here, we set $F^{h,\beta}(y) = \epsilon(f)h.y$. For the precise conventions on orientations and expressions, we refer to [YCC22].

- One can work out the algebra of the corresponding endomorphisms of $V(\Sigma, \Delta)$ and finds actions of the dual of the Drinfeld center. Its coproduct can be used to create ribbon operators Fh, β from elementary situations:



  In this way, we can create a pair consisting of an anyon and its conjugate at the two sites correspoonding to the end points of the ribbon. For more details, we refer to [YCC22].

1.2.4 Kuperberg invariant for three-manifolds

We briefly mention a related construction in quantum topology. Apart from the original work [Ku90], one good reference is [KV19]

Let V and W be handle bodies of equal genus g and $f : \partial V \to \partial W$ and orientation reversing homeomorphism. Gluing V to W along f yields a closed oriented 3-manifold

$$M = V \cup_f W \ .$$

Every closed, orientable three-manifold can be obtained in this way; this follows from results on the triangulability of three-manifolds

Definition 1.2.15 *The decomposition of M into two handlebodies is called a*

Heegaard splitting, and their common boundary Σ is called the Heegaard surface of the splitting.

The relevant information on the gluing along Σ can be encoded in terms of two sets of g circles, called upper and lower circles, on Σ. Circles of the same set are not allowed to intersect.

Order the circles; suppose that the i-th lower circle is intersected by n_i upper circles and the j-the upper circle by m_j lower circles. Note that

$$\sum_{i=1}^{g} n_i = \sum_{j=1}^{g} m_j =: N$$

equals the total number of intersection points. Consider

$$s_{lower} := \Delta^{n_1}(\Lambda) \otimes \ldots \otimes \Delta^{n_g}(\Lambda) \in H^{\otimes N}$$

and

$$s_{upper} := \Delta^{m_1}(\Lambda) \otimes \ldots \otimes \Delta^{m_g}(\Lambda) \in H^{*\otimes N}$$

The intersection of upper and lower circles relates tensorands in $H^{\otimes N}$ and $H^{*\otimes H}$. Evaluating according to this relation $\langle s_{upper}, s_{lower}\rangle$ when the tangent vectors of the lower and the upper circle at the intersection point are positively oriented and evaluating after applying the antipode S else, gives the Kuperberg invariant:

$$Z_{\text{Kup}}(H, M) = \langle s_{upper}, s_{lower}\rangle \ , .$$

It is interesting to note that a Hopf algebra H and its dual H^* give the same invariant. For more information, we refer to [Ku90].

Topological field theory links the (renormalized) Kuperberg invariant to the spaces of ground states of the Kitaev model, cf. remark 3.1.8.

1.3 Codes

There are two basic tasks in computing, both for classical and quantum computing:

- Storing information in a medium and transmitting information.
- Doing computations by processing information.

The first question leads to the mathematical notion of codes, the second to the notion of gates. We start our discussion with classical computing.

1.3.1 Classical codes

Implicitly, assumptions made on storage devices and manipulation of information in classical information theory are based on classical physics, as opposed to quantum mechanics.

Information is stored in the form of binary numbers, hence in terms of elements of the standard vector space $\mathbb{F}_2^n$ over the field $\mathbb{F}_2 = \{0, 1\}$ of two elements. (Note that the direct sum decomposition is crucial.) We identify the elements of $\mathbb{F}_2 = \{0, 1\}$ with either on/off or with the truth values $0 =$ false and $1 =$ true. If we are dealing with an element of $\mathbb{F}_2^n$, we say that we have n bits of information.

For storing and transmitting information, it is important that errors occurring in the transmission or by the dynamics of the storage device can be corrected. For this reason, only a subset $C \subset \mathbb{F}_2^n$ should correspond to valid information. This is also used in daily life, cf. e.g.

the composition of an IBAN or ISBN which contain checksum digits. (Look at the wikipedia pages!)

Definition 1.3.1

1. *A subset $C \subset (\mathbb{F}_2)^n$ is called a code. The natural number n is called the length of the code. One says that a code word $c \in C$ is composed of n bits.*

2. *We say that information is stored in the code, if an element $c \in C$ is selected.*

3. *A code $C \subset (\mathbb{F}_2)^n$ is called linear, if C is a vector subspace. Then $\dim_{\mathbb{F}_2} C =: k$ is called the dimension of the code.*

One can also allow instead of the field $\mathbb{F}_2$ an arbitrary finite field. We will not discuss this in more detail. To deal with error correction, one defines:

Definition 1.3.2

Let $K = \mathbb{F}_2$ and $V = K^n$. The map

$$d_H : V \times V \to \mathbb{N}$$
$$d_H(v, w) := |\{j \in \{1, \ldots, n\} | \, v_j \neq w_j\}|$$

is called Hemming distance. It equals the number of components (bits) in which the two code words v and w differ.

Lemma 1.3.3.

The Hemming distance has the following properties:

1. $d_H(v, w) \geq 0$ for all $v, w \in V$ and $d_H(v, w) = 0$, if and only $v = w$

2. $d_H(v, w) = d_H(w, v)$ for all $v, w \in V$ (symmetry)

3. $d_H(u, w) \leq d_H(u, v) + d_H(v, w)$ for all $u, v, w \in V$ (triangle inequality)

4. $d_H(v, w) = d_H(v + u, w + u)$ for all $u, v, w \in V$ (translation invariance)

Definition 1.3.4

For $\lambda \in \mathbb{N}$, a subset $C \subset (\mathbb{F}_2)^n$ is called a λ–error correcting code, if

$$d_H(u, v) \geq 2\lambda + 1 \quad \text{for all } u, v \in C \quad \text{with } u \neq v \ .$$

The reason for this name is the following

Lemma 1.3.5.

Let $C \subset V$ be a λ-error correcting code. Then for any $v \in V$, there is at most one $w \in C$ with $d_H(v, w) \leq \lambda$.

Proof.

Suppose we have $w_1, w_2 \in C$ with $d_H(v, w_i) \leq \lambda$ for $i = 1, 2$. Then the triangle inequality yields

$$d_H(w_1, w_2) \leq d_H(w_1, v) + d_H(v, w_2) \leq 2\lambda \ .$$

Since the code C is supposed to be λ-error correcting, we have $w_1 = w_2$. □

Remarks 1.3.6.

1. It is important to keep in mind the relative situation: a code C is a subspace of $\mathbb{F}_2^n$. The Hemming distance gives an indication to what extent the subspace C of code words is spread out in V.

2. If $C \subset \mathbb{F}_2^n$, we say that a codeword of C is composed of n bits. If C is a linear code with $\dim_{\mathbb{F}_2} C = k$, we refer to a $[n, k]$ code. Denote by

$$d := \min_{c \in C \setminus \{0\}} d_H(c, 0)$$

the minimal distance of a code. We refer to an $[n, k, d]$ code. In practice, the length n of the code has to be kept small, because this causes costs for storing and transmitting. The minimal distance d has to be big, since by lemma 1.3.5 this allows to many correct errors. The dimension k of the code has to be big enough to allow enough code words. From elementary linear algebra, one derives the singleton bound

$$k + d \leq n + 1$$

which shows that these goals are in competition.

Remarks 1.3.7.

1. Classical storage devices are typically localized, either in space (e.g. an electron or a nuclear spin) or in momentum space (e.g. a photon polarization).

2. Many storage devices are magnetic, i.e. a collection of coupled spins. The Hamiltonian is such that it favours the alignment of spins. So if one spin is kicked out by thermal fluctuation, the Hamiltonian tends to push it back in the right position. Thus errors in the storage device are corrected by the dynamics of the system; this is modelled in the Kitaev model.

1.3.2 Classical gates

To process information, we need logical gates: A logical gate takes as an input n bits of information and yields m bits as an output.

Definition 1.3.8
Let $K = \mathbb{F}_2$.

1. *A gate is map $f : K^n \to K^m$. Typically, one requires a gate to act non-trivially only on few, two or three) bits, i.e. to act as the identity on all except for a few summands of K^n.*

2. *A gate is called linear, if the map f is K-linear.*

3. *If the map f is invertible, the gate is called reversible.*

4. *A finite set of gates is called a library of gates. One then applies to $\mathbb{F}_2^n$ a sequence of gates in the library acting on any subset of summands in $\mathbb{F}^n$ and as the identity elsewhere. The composition of such maps is called a circuit.*

5. *A library of gates is called universal, for any Boolean function $f(x_1, x_2, ..., x_m)$, there is a circuit consisting of gates in the library which takes $x_1, x_2, \ldots, x_m$ and some extra bits set to 0 or 1 and outputs $x_1, x_2, \ldots, x_m, f(x_1, x_2, \ldots, x_m)$, and some extra bits (called garbage). Essentially, this means that one can use the gates in the library to build systems that perform any desired Boolean function computation.*

We wish to use gates to implement the basic Boolean operations:

Examples 1.3.9.

1. Basic gates include negation NOT, AND and OR:

A	$\neg A$
t	f
f	t

A	B	$A \wedge B$
t	t	t
t	f	f
f	t	f
f	f	f

A	B	$A \vee B$
t	t	t
t	f	t
f	t	t
f	f	f

These are gates acting on one bit resp. mapping two bits to one bit.

2. Also in use are the following gates acting on two bits:

A	B	NAND
t	t	f
t	f	t
f	t	t
f	f	t

A	B	NOR
t	t	f
t	f	f
f	t	f
f	f	t

A	B	XOR
t	t	f
t	f	t
f	t	t
f	f	f

3. It is an important theoretical question whether a library of gates is universal. For example, the NAND gate is universal:

   - To get the NOT gate, double the input and feed it into a NAND gate.
   - To get the AND gate, take a NAND gate, followed by a NOT gate, which can be constructed from a NAND gate.
   - To get an OR gate, use de Morgan's law: apply NOT gates to both inputs and feed it into a NAND gate.

4. To add two bits A and B, double the bits and feed them into a XOR gate to get the last digit S of the sum and into an AND gate to get a carry-on bit C:

A	B	S=XOR	C=AND
t=1	t=1	f=0	t=1
t=1	f=0	t=1	f=0
f=0	t=1	t=1	f=0
f=0	f=0	f=0	f=0

In such a way, one realizes the arithmetic operations on natural numbers.

1.3.3 Quantum computing

Quantum computation is based on quantum mechanical systems.

- States can be superposed, which leads to a linearization.
- The uncertainty principle introduces new limitations, e.g. quantum information cannot be copied: there is no complete set of observables characterizing a state completely that can be measured simultaneously. There is no canonical linear map $V \to V \otimes V$ that can be defined without choosing additional structure on V.

The Kitaev model realizes some aspects of a quantum mechanical system in a direct way. We would get even closer to the standard setting of quantum mechanics using Hilbert spaces using Hopf-$*$-algebras, from which we refrain in these lectures. Consider the case of the group algebra $H = K[\mathbb{Z}_2]$.

- We can interpret the degrees of freedom as a spin 1/2-particle with basis vectors $b_e = |\uparrow\rangle$ and $b_g| \downarrow\rangle$.
- The analogue of the ambient vector space $\mathbb{F}_2^n$ is the space of ground states $Z_K(H;\Sigma,\Delta)$ or, more generally, the space of protected states $Z_K(H;\Sigma,Y_1,\ldots Y_n)$. It does not create problems that it depends on the discrete structure Δ.

  If this space has dimension n, we will say that a code vector v in it is composed of n qubits.
- The dynamics of the Kitaev model is designed to ensure some error correction.

To get a framework for quantum computing, we use the following aspects of the Kitaev model:

- Codes, i.e. interesting subspaces of an ambient vector space. (In the literature, this space is typically taken to be finite-dimensional.) They should have properties to make the system fault tolerant.
- Quantum gates, i.e. (unitary) operators acting on $H^{\otimes n}$ that preserve these subspaces, are created by ribbon operators.

1.3.4 Some definitions

For quantum gates, we need unitary operators $H^{\otimes n} \to H^{\otimes n}$ to be realized by some time evolution in pracize or ribbon operators in the Kitaev model:

Definition 1.3.10

1. *A quantum gate on $H^{\otimes n}$ is a unitary map $H^{\otimes n} \to H^{\otimes n}$ that acts as the identity on at least $n-2$ tensorands.*
2. *Consider a fixed finite set $\{U_i\}_{i\in I}$ of quantum gates, i.e. $U_i \in \mathrm{U}(H)$ or $U_i \in \mathrm{U}(H\otimes H)$, called a library of quantum gates. Denote by $U_i^{\alpha\beta}$ the gate U_i acting on the α and β tensorand resp. U_i^α acting on the α tensorand of H^n. A quantum circuit based on this library is a finite product of U_i^α and $U_i^{\alpha\beta}$. It is a unitary endomorphism of $H^{\otimes n}$.*
3. *A library of quantum gates is called universal, if for any n, the subgroup of $\mathrm{U}(H^{\otimes n})$ generated by all circuits is dense.*

For quantum codes, we refer to [FKLW03] as a general reference:

Definition 1.3.11

1. *A quantum code is a linear subspace $W \subset V = H^{\otimes n}$ of a quantum medium V. A quantum code is also called a protected space.*
2. *Let $0 \le k \le n$. A k-local operator is a linear map $O: V \to V$ which is the identity on $n-k$ tensorands of V. (By definition 1.3.10, quantum gates are thus at least 2-local.)*
3. *Denote by $\pi_W: V \to W$ the orthogonal projection. A quantum code $W \subset V$ is called a k-code, if the linear operator*
$$\pi_W \circ O: \quad W \to W$$
is multiplication by a scalar for any k-local operator O.

One can show the following analogue of a lemma 1.3.5:

Lemma 1.3.12.
If W is a k-code, then information cannot be degraded from errors operating on less than $\frac{k}{2}$ of the n particles.

Remarks 1.3.13.

1. A first attempt to realize qubits might be to take *isolated* trapped particles, individual atoms, trapped ions or quantum dots. Such a configuration is fragile and one has to minimize any external interaction. On the other hand, external interaction is needed to write and read off information.

   The idea of topological quantum computing is to use non-local degrees of freedom to produce fault tolerant subspaces. Concretely, one needs non-abelian anyons in quasi two-dimensional systems.

2. Storage devices are typically effectively two-dimensional. Thus the protected subspace should be the space of states of a three-dimensional topological field theory. Maps describing gates and circuits are obtained from colored cobordisms, i.e. three-manifolds containing links.

3. A theorem of Freedman, Kitaev and Wang asserts that quantum computers and classical computers can perform exactly the same computations. But their efficiency is different, e.g. for problems like factoring integers into primes.

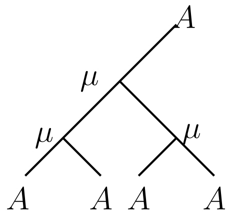
2 Lecture 2: Pivotal categories and string net models

The finite tensor category $H-$mod given by finite-dimensional modules over a Hopf algebra H is rather special as a monoidal category. Interesting categories (representation categories e.g. of loop groups or vertex algebras) cannot be written in this form. We thus need other tools.

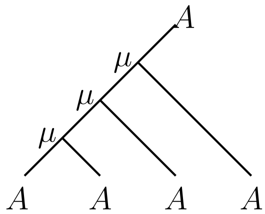
2.1 Graphical calculus for pivotal bicategories

2.1.1 Bicategories

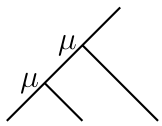
Some mathematical theories can be described in terms of a graphical calculus, for example multiple multiplications in an associative algebra in terms of trees:



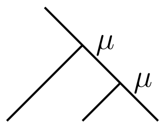
or



- The tree is labeled: edges by the algebra A and vertices by the multiplication μ
- Trees can be evaluated to a map of vector spaces, in the examples to maps $A^{\otimes 4} \to A$.
- The local relation

  $\leftrightarrow$

  can be applied locally inside any tree and relates different trees with same evaluation. Interpret it as a pair of mutually inverse arrows on the set of trees with same endpoints.

Our graphical calculus uses as an input bicategories. Categories are a two-layered structure, comprising objects and morphisms. Bicategories are a three-layered structure, comprising objects, 1-morphisms and 2-morphisms. The first examples are

- The bicategory Cat. Objects are categories $\mathcal{C}, \mathcal{D}$, 1-morphism functors $F, G : \mathcal{C} \to \mathcal{D}$ and 2-morphisms natural transformations $\alpha : F \Rightarrow G$.
- Any K-algebra A gives rise to a category $*//A$ with one object and morphisms given by elements of A; composition comes from the product of A. Similarly, any monoidal category $\mathcal{C}$ gives rise to a one-object bicategory $*//\mathcal{C}$ with 1-morphisms the objects of $\mathcal{C}$ and 2-morphisms the morphisms of $\mathcal{C}$.

Definition 2.1.1
A *bicategory* $\mathcal{B}$ *consists of the following data subject to the following axioms. The data are*

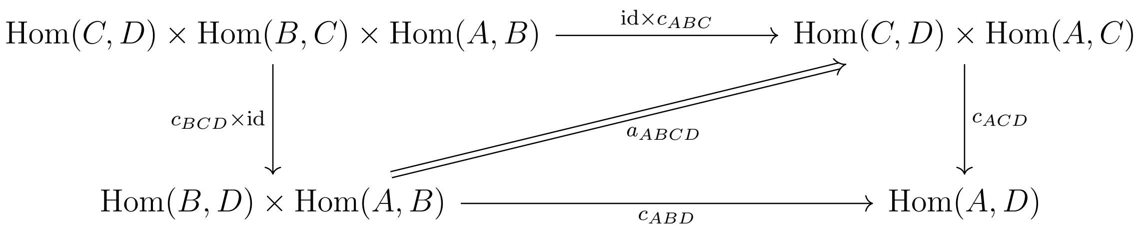
- *A class* ob $\mathcal{B}$ *with elements* $A, B, \dots$ *which we depict as 0-cells.*
- *Categories* $\mathrm{Hom}(A, B)$ *for each pair* $A, B \in$ ob $\mathcal{B}$, *whose objects* f, g *we call a 1-cells or 1-morphisms and whose arrows* $\alpha, \beta, \dots$ *we call 2-cells or 2-morphisms.*

- *Composition functors for any triple* $A, B, C \in \text{ob}\,\mathcal{B}$

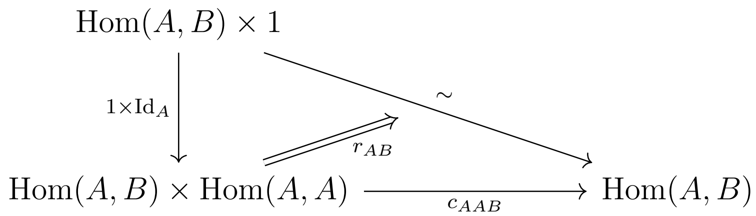
$$\begin{array}{rrcl} c_{ABC}: & \text{Hom}(B,C)\times\text{Hom}(A,B) & \to & \text{Hom}(A,C) \\ & (g,f) & \mapsto & g\circ f \\ & (\beta,\alpha) & \mapsto & \beta\circ\alpha \end{array}$$

and an identity functor $\text{Id}_A : 1 := *//\text{id}_* \to \text{Hom}(A,A)$ *for any object* $A \in \text{ob}\,\mathcal{B}$*. Note that this gives for each object* A *of a bicategory an identity 1-morphism* Id_A *and an identity 2-morphism* $\text{Id}_A \to \text{Id}_A$*.*

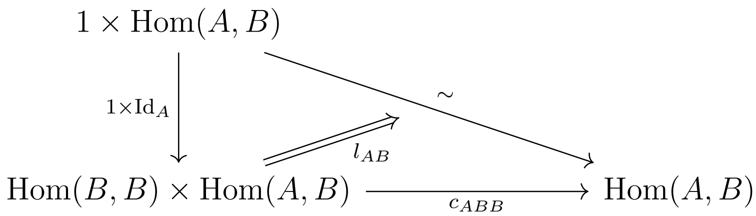
- *Natural isomorphisms* a, r, l *of functors expressing associativity*



and unitality:



and

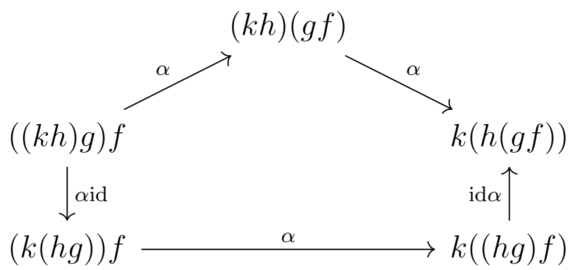


thus 2-cells

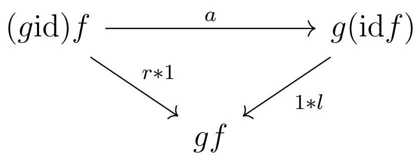
$$\begin{array}{rccc} a_{hgf}: & (hg)f & \tilde{\to} & h(gf) \\ r_f: & f\circ I_A & \tilde{\to} & f \\ l_f: & I_B\circ f & \tilde{\to} & f. \end{array}$$

Axioms: the following diagrams with arrows labelled by 2-morphisms commute:

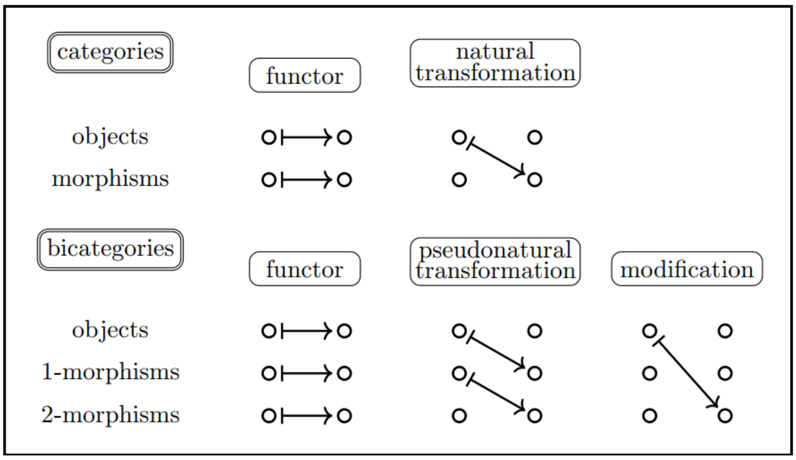
- *Pentagon diagrams*



- *Triangle diagrams*

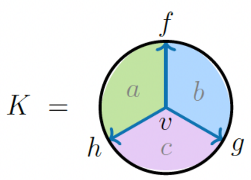


We should now also define 2-functors, pseudonatural transformations and modifications. We should now proceed and introduce the notion of a (symmetric) monoidal bicategory. We summarize some aspects in the following scheme:



2.1.2 Graphical calculus

The input datum for the graphical calculus is a linear bicategory $\mathcal{B}$. For our graphical calculus, (partially) $\mathcal{B}$-labelled corollas are the basic objects.



(The empty set is also a corolla.)

The corollas are labelled:

- 2-cells are labelled by objects a, b, c of a bicategory $\mathcal{B}$.
- 1-cells are then labelled by 1-morphisms f, g, h of a bicategory $\mathcal{B}$. Source and target are determined by the orientations.
- No label for the center v of the corolla.

In the Kitaev construction, the fact that $S^2 = \mathrm{id}_H$ was heavily used. It implies that for any H-module M, the identity provides an isomorphism $M^{\vee\vee} \to M$.

Definition 2.1.2

1. *Let $\mathcal{C}$ be a right rigid monoidal category. A* *pivotal structure* *is a monoidal natural isomorphism*

$$\omega : \quad \mathrm{id}_{\mathcal{C}} \to ?^{\vee\vee} \ .$$

A right rigid monoidal category together with a choice of pivotal structure is called a *pivotal category*.

2. *A* *pivotal Hopf algebra* *is a pair (H, ω), where H is a Hopf algebra and $\omega \in G(H)$ is a group-like element, called the* *pivot* *such that*

$$S^2(x) = \omega x \omega^{-1} \quad \textit{for all } x \in H \ .$$

3. *A pivotal structure on a bicategory $\mathcal{B}$ with fixed left and right duals is an identity component pseudonatural transformation (i.e. every component 1-morphism is an identity) $\mathrm{id}\mathcal{B} \Rightarrow (-)^{\vee\vee}$. Equipped with a pivotal structure, $\mathcal{B}$ is called a pivotal bicategory.*

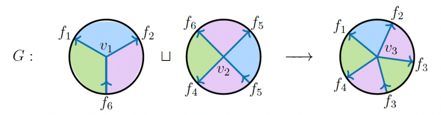
4. *A strictly pivotal bicategory is a pivotal bicategory for which the double dual is the identity.*

In particular, for a semisimple Hopf algebra H, the category H-mod comes with a canonical pivotal structure given by $\omega = 1_H$. Pivotal structures can be strictified.

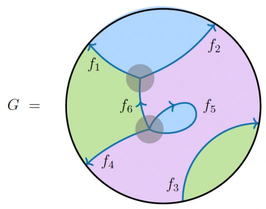
A pivotal structure allows us to reverse the orientation of a 1-cell in a corolla by replacing at the same time a 1-morphism by its dual.

Definition 2.1.3 *Given a pivotal bicategory, we define a monoidal category* $\mathrm{Corollas}_{\mathcal{B}}^{\sqcup}$.

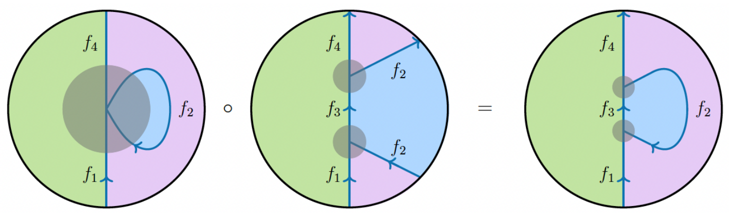
- *Objects are disjoint unions of $\mathcal{B}$-corollas.*
- *Morphisms are generated by $\mathcal{B}$-labeled discs. For example, a morphism*



is given by the disc



- *Composition is given by insertion of discs:*



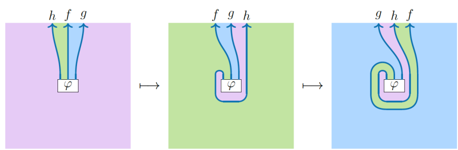
- *The monoidal structure is given by finite disjoint unions.*

Proposition 2.1.4. Any linear pivotal bicategory $\mathcal{B}$ comes with a symmetric monoidal functor

$$\mathrm{GCal}_{\mathcal{B}} : \ \mathrm{Corollas}_{\mathcal{B}}^{\sqcup} \longrightarrow \mathrm{vect}_k^{\otimes}$$

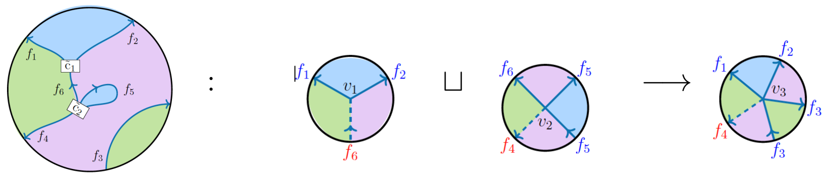
which we call its graphical calculus.

Proof. • The following picture should suggest that a pivotal bicategory comes with cyclically invariant tensors:

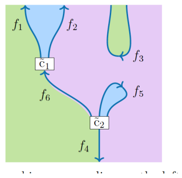


The functor $\mathrm{GCal}_{\mathcal{B}}$ maps a corolla to the vector space of cyclically invariant tensors.

- We also give a linear map for the morphism of corollas:



Pick polarizations and consider the morphism given by the evaluation of progressive diagram in the pivotal bicategory:



(Here, we are using quite a few facts: for pivotal monoidal categories, the Reshetikhin-Turaev functors allows to evaluate progressive planar diagrams. This can be extended to pivotal bicategories.)

□

2.2 String-net models, skein theory

The motto is: "String net modular functor is a globalized version of graphical calculus". Since we want to construct open/closed modular functors, we assume that our pivotal bicategory is pointed, i.e. that there is a distinguished object.

2.2.1 String-net vector spaces

Fix an oriented surface Σ, possibly with boundaries or even corners. Fix a boundary value:

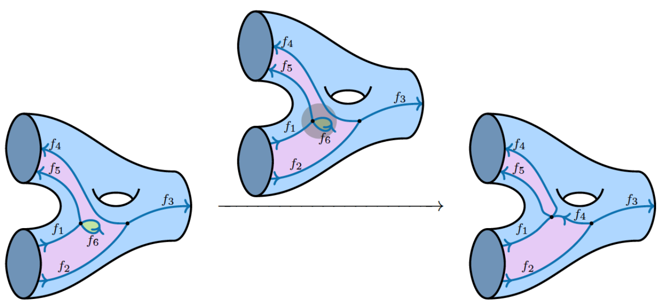
- For a circle $\mathbb{S}^1$, these are finitely many oriented points which partition the circle into intervals. The intervals are labelled by objects of $\mathcal{B}$, the points by appropriate 1-morphisms.

We now set up an evaluation functor

$$\mathcal{E}_{\mathcal{B}}^{\Sigma,b}: \quad \mathcal{G}\text{raphs}_{\mathcal{B}}(\Sigma, b) \to \text{vect}_k$$

- Objects of $\mathcal{G}\text{raphs}(\Sigma, b)$ are partially $\mathcal{B}$-colored finite two-dimensional CW structures on Σ: 2-cells are colored with objects of $\mathcal{B}$, 1-cells with 1-morphisms. Vertices are *not colored*. The graphs under consideration have the prescribed boundary values b.

- Morphisms of $\mathcal{G}\text{raphs}(\Sigma, b)$ are freely generated discs on Σ which intersect a graph transversally



The evaluation functor sends the this morphism in $\mathcal{G}\mathrm{raphs}(\Sigma, b)$ to the linear map

$$\bigotimes_{v\in V(\Gamma_1)} H_v^{\mathcal{B}} \longrightarrow \bigotimes_{v\in V(\Gamma_2)} H_v^{\mathcal{B}}$$

where we use the fact that the tensor product in vect_K is symmetric to define the objects and where the arrow is given by the linear map from the graphical calculus.

Definition 2.2.1 *The string net space associated to* (Σ, b) *is the colimit* $\mathrm{sn}^{\circ}_{\mathcal{B}}(\Sigma, b) = \mathrm{colim}\ \mathcal{E}_{\mathcal{B}}^{\Sigma,b}$

Explicitly, we take the vector space, freely generated by partially $\mathcal{B}$-colored finite two-dimensional CW-complexes on Σ and identify two linear combinations, if they coincide outside a disc on Σ and evaluate to the same value under the graphical calculus. We have here such general conditions that we cannot expect the string net vector space to be finite-dimensional.

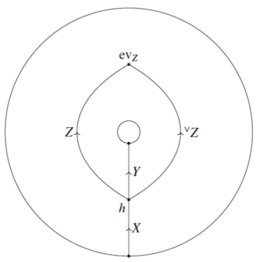
Examples 2.2.2.

1. For any spherical fusion category $\mathcal{C}$, the string-net space associated to the sphere $\mathbb{S}^2$ is $\mathrm{End}_{\mathcal{C}}(1) \cong K\mathrm{id}_{\mathbb{I}}$ and spanned by the empty graph.

2. Assume for simplicity that $\mathcal{C}$ is a spherical fusion category. For the torus $\mathbb{T}^2$, we find

$$\mathrm{sn}(\mathbb{T}^2) = \oplus_{X,Y}\mathrm{Hom}(X, Y \otimes X \otimes {}^{\vee}Y)\ ,$$

where the sum is over isomorphism classes of simple objects of $\mathcal{C}$:



For a finite abelian group A, consider the group algebra $K[A]$ and $\mathcal{C} = H-\mathrm{mod}$. Then we obtain an $|A|^2$-dimensional vector space. This shows that the dimension of the string-net space is sensitive to the topology of Σ.

Remarks 2.2.3.

1. The mapping class group $\mathrm{MCG}(\Sigma)$ acts on the string net space.

2. The first part gives us quantum gates.

2.2.2 Double categories

We now want to define a modular functor. It is convenient to go beyond bicategories:

Definition 2.2.4 *A double category* $\mathbb{A}$ *consists of the following data:*

- *a collection of* objects $a, b, c, ... \in \mathbb{A}$*;*
- *a collection of* vertical 1-morphisms $f: a \to\ b$, $g: b \to\ c, ...$
- *a composition of vertical* 1*-morphisms;*

- *a collection of* horizontal 1-morphisms $P\colon a \nrightarrow b$, $Q\colon b \nrightarrow c, \ldots$
- *a composition of horizontal* 1*-morphisms;*
- *a collection of* 2-morphisms $\begin{array}{ccc} a & \overset{P}{\nrightarrow} & b \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\ a' & \underset{Q}{\nrightarrow} & b' \end{array}$, ...
- *vertical and horizontal compositions of* 2*-morphisms, e.g. (suppressing labels of 1-morphisms)*

$$\begin{array}{ccc} a & \nrightarrow & b \\ \Big\downarrow & \alpha' \circ \alpha & \Big\downarrow \\ a'' & \nrightarrow & b'' \end{array} \quad = \quad \begin{array}{ccc} a & \nrightarrow & b \\ \downarrow & \alpha & \downarrow \\ a' & \nrightarrow & b' \\ \downarrow & \alpha' & \downarrow \\ a'' & \nrightarrow & b'' \end{array} \tag{7}$$

and

$$\begin{array}{ccc} a & \longrightarrow\!\!\!\!\!\!\!\!\!+\quad\;\; & c \\ \downarrow & \alpha \cdot \beta & \downarrow \\ a' & \longrightarrow\!\!\!\!\!\!\!\!\!+\quad\;\; & c' \end{array} \quad = \quad \begin{array}{ccccc} a & \nrightarrow & b & \nrightarrow & c \\ \downarrow & \alpha & \downarrow & \beta & \downarrow \\ a' & \nrightarrow & b' & \nrightarrow & c' \end{array} \tag{8}$$

The composition of vertical 1*-morphisms, which we denote by '$\circ$', is strictly unital and associative, while the composition of horizontal* 1*-morphisms, to be denoted by '$\cdot$', is weakly unital and associative.*
Similarly, the vertical composition of 2-morphisms is strictly unital and associative, while their horizontal composition is unital and associative only up to coherent isomorphisms.

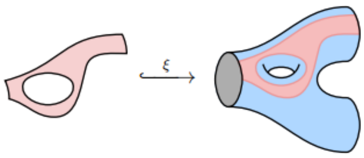
Moreover the two types of compositions are compatible, as expressed by a middle-four interchange law *akin to the one for bicategories.*

Example 2.2.5. There is a symmetric monoidal double category $\mathbb{OCB}\mathrm{ord}_2^{\mathrm{or}}$ with the following data:

- objects: compact oriented one-manifolds with possibly non-empty boundary;
- vertical 1-morphisms: orientation preserving smooth embeddings; they incorporate an important notion of locality in approaches to field theory.
- horizontal 1-morphisms: oriented open-closed bordisms, with composition given by sewing;
- 2-morphisms: isotopy classes of orientation preserving embeddings that restrict to the embeddings of the parametrizing one-manifolds. For example, there is a 2-morphism

$$\begin{array}{ccc} \mathbb{I} & \overset{\Sigma}{\nrightarrow} & \mathbb{I} \\ {\scriptstyle f}\downarrow & \xi & \downarrow{\scriptstyle g} \\ \mathbb{S}^1 & \underset{\Sigma'}{\nrightarrow} & (\mathbb{S}^1)^{\sqcup 2} \end{array} \tag{9}$$

that looks like



- monoidal product: disjoint union.

Remark 2.2.6. 1. To any double category $\mathbb{A}$, we can associate a bicategory $\mathcal{H}(\mathbb{A})$: it has the same objects as $\mathbb{A}$, the horizontal 1-morphisms of $\mathbb{A}$ as 1-morphisms and 2-morphisms of the form

$$\begin{array}{ccc} a & \overset{P}{\nrightarrow} & b \\ \mathrm{id}\downarrow & \alpha & \downarrow\mathrm{id} \\ a' & \underset{Q}{\nrightarrow} & b' \end{array}$$

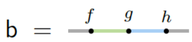
2. Under precise conditions, see [WHS19], a monoidal double category $\mathbb{A}$ gives rise to monoidal bicategory $\mathcal{H}(\mathbb{A})$. In the case of cobordisms, this gives the standard symmetric monoidal cobordism bicategory $\mathcal{H}(\mathbb{OC}\mathrm{Bord}_2^{\mathrm{or}}) = \mathcal{B}\mathrm{ord}_2^{\mathrm{or,o/c}}$.

2.2.3 Cylinder categories

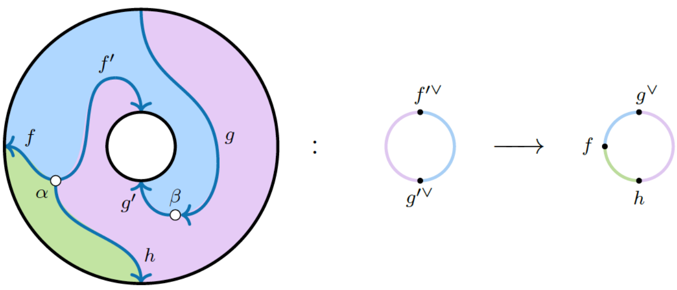
Fix a pointed pivotal bicategory $\mathcal{B}$.

Definition 2.2.7 *For any closed oriented 1-manifold ℓ, define a $\mathbb{K}$-linear category $\mathrm{Cyl}^\circ(\mathcal{B}, \ell)$. Objects are $\mathcal{B}$-boundary values, morphisms are string-net spaces on a cylinder (=annulus). Composition is stacking.*

For 1-manifolds with boundary, use the distinguished object $*_{\mathcal{B}} \in \mathcal{B}_0$ as a label for the 1-cell adjacent to a boundary point (in grey):



Example 2.2.8. For instance, for $\ell = \mathbb{S}^1$ and any choice of morphisms α and β,



is a morphism in $\mathrm{Cyl}^\circ(\mathcal{C}, \mathbb{S}^1)$.

Remarks 2.2.9. 1. We have functoriality under embedding of 1-manifolds: symmetric monoidal functor $\mathrm{Cyl}^\circ(\mathcal{B}, *_{\mathcal{B}}, -) : \mathrm{Emb}_1^{or} \to \mathrm{Cat}_K$

2. We have $\mathrm{Cyl}^\circ(\mathcal{B}, a, I) \cong \mathcal{B}(a, a)$. In the special case $\mathcal{B} = *//\mathcal{C}$ with $\mathcal{C}$ pivotal fusion category, we get $\mathrm{Cyl}(\mathcal{C}, I) \cong \mathcal{C}$.

3. Using the description of string net spaces on a cylinder from examples 2.2.2, we find, if $\mathcal{C}$ is a spherical fusion category

$$\mathrm{Hom}_{\mathrm{Cyl}^\circ(\mathcal{C},\mathbb{S}^1)}(X,Y) = \oplus_Z \mathrm{Hom}_\mathcal{C}(X, Z\otimes Y \otimes {}^\vee Z) \cong \mathrm{Hom}_\mathcal{C}(X, \oplus_Z Z\otimes Y\otimes {}^\vee Z)$$

where the sum is over isomorphism classes of simple objects of $\mathcal{C}$.

4. The forgetful functor $U : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ which forgets the half-braiding has a left adjoint $I \dashv U$ which for the case when $\mathcal{C}$ is a fusion category gives the object

$$I(c) := \oplus_Z Z \otimes c \otimes {}^\vee Z$$

of $\mathcal{C}$ with a certain half-braiding. Thus

$$UI(c) = \oplus_Z Z \otimes c \otimes {}^\vee Z \ .$$

We thus find for the cylinder category

$$\mathrm{Hom}_{\mathrm{Cyl}^\circ(\mathcal{C},\mathbb{S}^1)}(X,Y) \cong \mathrm{Hom}_\mathcal{C}(X, UI(Y)) \cong \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(IX, IY) \ .$$

We have thus recovered (a full subcategory of the) Drinfeld center from the string-net construction, the Kleisli category of the central monad. For more information, we refer to [Ki11, Section 6]. (Note that for $\mathcal{C} = H$-mod, the Drinfeld center is $D(H)$-mod.)

Note that for two boundary values, we get a vector space. This applies to general surfaces as well. Moreover, one checks from the definition of the cylinder category that reversing the orientation gives the opposite category. Hence a cobordism

$$\begin{array}{ccc} & M & \\ \nearrow & & \nwarrow \\ \Sigma_1 & & \Sigma_2 \end{array}$$

gives a functor

$$\mathrm{sn}_\mathcal{B} : \quad \mathrm{Cyl}^\circ(\mathcal{B},\Sigma_1) \otimes \mathrm{Cyl}^\circ(\mathcal{B},\Sigma_2)^{opp} \to \mathrm{vect}_K \ .$$

This leads us to consider profunctors:

Example 2.2.10. For any field K there is a symmetric monoidal double category $\mathbb{P}\mathrm{rof}_K$ with the following defining data:

- objects: essentially small K-linear categories;
- vertical 1-morphisms: K-linear functors;
- horizontal 1-morphisms: K-linear profunctors. (A profunctor $P\colon A \nrightarrow B$ is a functor $A \otimes B^{\mathrm{opp}} \to \mathrm{vect}$, with composition given by coends.);
- 2-morphisms: natural transformations, with functors inserted in the target profunctor; for instance, the 2-morphism

$$\begin{array}{ccc} A & \xrightarrow{P} & B \\ {\scriptstyle F}\downarrow & \varphi & \downarrow{\scriptstyle G} \\ A' & \xrightarrow[Q]{} & B' \end{array} \tag{10}$$

is given by a natural transformation

$$\varphi\colon P(-,\sim) \Longrightarrow Q(F(-),G(\sim)). \tag{11}$$

- monoidal product: the naive tensor product for enriched categories.

Again, we recover the ordinary (symmetric monoidal) bicategory of profunctors by $\mathcal{P}\mathrm{rof}_K = \mathcal{H}(\mathbb{P}\mathrm{rof}_K)$.

Any functor $F : \mathcal{A} \to \mathcal{B}$ between linear categories gives rise to two profunctors

$$\begin{array}{llllllll} F_* : & \mathcal{A}^{opp} \times \mathcal{B} & \to & \mathrm{vect} & (a,b) & \mapsto & \mathrm{Hom}_{\mathcal{B}}(Fa, b) & \mathcal{B} \nrightarrow \mathcal{A} \\ F^* : & \mathcal{B}^{opp} \times \mathcal{A} & \to & \mathrm{vect} & (a,b) & \mapsto & \mathrm{Hom}_{\mathcal{B}}(b, Fa) & \mathcal{A} \nrightarrow \mathcal{B} \end{array}$$

2.2.4 The modular functors

We then state [FSY26, Theorem 5.1]:

Theorem 2.2.11. Let $(\mathcal{B}, *_{\mathcal{B}})$ be a pointed pivotal bicategory. There is a symmetric monoidal double functor, a modular functor,

$$\mathbb{SN}_{\mathcal{B}} \colon \mathbb{OCB}\mathrm{ord}_2^{\mathrm{or}} \to \mathbb{P}\mathrm{rof}_K \tag{12}$$

In particular, factorization holds: the functor $\mathrm{sn}^0_{\mathcal{B}}(\cup_\beta \Sigma; -, \sim)$ can be endowed with the structure of a coend

$$\int^{b \in \mathrm{Cyl}^0(\mathcal{B},\beta)} \mathrm{sn}^0_{\mathcal{B}}(-, b, b, \sim) : \quad \mathrm{Cyl}^0(\mathcal{B}, *_{\mathcal{B}}, \alpha) \to \mathrm{Cyl}^0(\mathcal{B}, *_{\mathcal{B}}, \gamma)$$

Remarks 2.2.12.

1. By similar considerations as those mentioned in remark 2.2.6, the symmetric monoidal double functor gives rise to a symmetric monoidal functor of bicategories:

$$\mathrm{SN}_{\mathcal{B}} \colon \mathcal{B}\mathrm{ord}_2^{\mathrm{or,o/c}} \to \mathcal{P}\mathrm{rof}_K$$

2. We have $\mathrm{SN}_{\mathcal{B}}(\emptyset) = \mathrm{vect}$, where $\ell = \emptyset$ is considered as a one-manifold. To a closed oriented surface, we assign a profunctor vect $\nrightarrow$ vect, i.e. a vector space. For $\mathcal{C} = H$-mod and $\mathcal{B} = *//\mathcal{C}$, this functor assigns to a closed oriented surface the vector space we constructed with the Kitaev construction.

3. The vector spaces assigned to surfaces now come with actions of mapping class groups, allowing for the implementation of quantum gates.

4. If $\mathcal{C}$ is a pivotal fusion category, then the modular functor can be extended to a three-dimensional TFT, see [B22].

2.3 Remarks on CFT correlators

One application of the string net construction is a mathematical understanding of CFT correlators. This section is based on [FSY26]. Let $\mathcal{C}$ be a pivotal linear tensor category. (Readers directly interested in conformal field theory may wish to imagine $\mathcal{C}$ as a category of representations of a suitably nice vertex operator algebra, ignoring the braiding.)

The monoidal category $\mathcal{C}$ gives rise to two bicategories:

- On the one hand, $*//\mathcal{C}$, a bicategory with a single object whose 1-morphisms are the objects in $\mathcal{C}$ and whose 2-morphisms are the morphisms in $\mathcal{C}$.
- And on the other hand, the bicategory $\mathcal{F}r(\mathcal{C})$ of Δ-separable symmetric Frobenius algebras in $\mathcal{C}$, with bimodules as 1-morphisms.

Both $*//\mathcal{C}$ and $\mathcal{F}r(\mathcal{C})$ are pivotal, and both are pointed, i.e. endowed with a distinguished object. For $*//\mathcal{C}$ the latter is, obviously, the single object $*$, while for $\mathcal{F}r(\mathcal{C})$ the distinguished object is the Frobenius algebra defined on the monoidal unit $\mathbb{I}_{\mathcal{C}}$.

There is a forgetful functor

$$u\colon \quad \mathcal{F}r(\mathcal{C}) \Rightarrow \ * //\mathcal{C}$$

sending any Frobenius algebra to $*$, a bimodule $B : A_1 \to A_2$ to its underlying object in $\mathcal{C}$ and a morphism of bimodules to the morphism in $\mathcal{C}$. It has the structure of a separable Frobenius functor. This functor relates the graphical calculi and thus the string net modular functors.

Here, relating means:

Definition 2.3.1 *Let $F, G\colon \mathbb{A} \to \ \mathbb{B}$ be two double functors. A* <u>*vertical transformation*</u> *(or* <u>*double transformation*</u>*) $\theta\colon F \Longrightarrow G$ consists of two pieces of data: a family*

$$\{\theta_a\colon Fa \Rightarrow \ Ga\}_{a\in\mathbb{A}} \tag{13}$$

of vertical 1-morphisms in $\mathbb{B}$, called the components of the double transformation at objects, and a family

$$\begin{array}{ccc} Fa & \xrightarrow{FP} & Fb \\ {\scriptstyle\theta_a}\big\downarrow & \theta_P & \big\downarrow{\scriptstyle\theta_b} \\ Ga & \xrightarrow[GP]{} & Gb \end{array} \tag{14}$$

of 2-morphisms in $\mathbb{B}$ that is parametrized by the horizontal 1-morphisms $P\colon a \nrightarrow b$ in $\mathbb{A}$. These data are required to satisfy the following conditions:

- Horizontal functoriality*:*

$$\begin{array}{ccccc} Fa & \xrightarrow{FP} & Fb & \xrightarrow{FQ} & Fc \\ {\scriptstyle\theta_a}\big\downarrow & \theta_P & \big\downarrow{\scriptstyle\theta_b} & \theta_Q & \big\downarrow{\scriptstyle\theta_c} \\ Ga & \xrightarrow[GP]{} & Gb & \xrightarrow[GQ]{} & Gc \end{array} \;=\; \begin{array}{ccc} Fa & \xrightarrow{F(P\cdot Q)} & Fc \\ {\scriptstyle\theta_a}\big\downarrow & \theta_{P\cdot Q} & \big\downarrow{\scriptstyle\theta_c} \\ Ga & \xrightarrow[G(P\cdot Q)]{} & Gc \end{array} \tag{15}$$

- Vertical naturality*:*

$$\begin{array}{ccc} Fa & \xrightarrow{FP} & Fb \\ {\scriptstyle Ff}\big\downarrow & F\alpha & \big\downarrow{\scriptstyle Fg} \\ Fa' & \xrightarrow[FP']{} & Fb' \\ {\scriptstyle\theta_{a'}}\big\downarrow & \theta_{P'} & \big\downarrow{\scriptstyle\theta_{b'}} \\ Ga' & \xrightarrow[GP']{} & Gb' \end{array} \;=\; \begin{array}{ccc} Fa & \xrightarrow{FP} & Fb \\ {\scriptstyle\theta_a}\big\downarrow & \theta_P & \big\downarrow{\scriptstyle\theta_b} \\ Ga & \xrightarrow[GP]{} & Gb \\ {\scriptstyle Gf}\big\downarrow & G\alpha & \big\downarrow{\scriptstyle Gg} \\ Ga' & \xrightarrow[GP']{} & Gb' \end{array} \tag{16}$$

  for any 2-morphism α.

- Horizontal unitality*:*

$$\begin{array}{ccc} Fa & \xrightarrow{F(U_a)} & Fa \\ {\scriptstyle\theta_a}\big\downarrow & \theta_{U_a} & \big\downarrow{\scriptstyle\theta_a} \\ Ga & \xrightarrow[G(U_a)]{} & Ga \end{array} \;=\; \begin{array}{ccc} Fa & \xrightarrow{U_{Fa}} & Fa \\ {\scriptstyle\theta_a}\big\downarrow & U_{\theta_a} & \big\downarrow{\scriptstyle\theta_a} \\ Ga & \xrightarrow[U_{Ga}]{} & Ga \end{array} \tag{17}$$

One can prove the following theorem [FSY26, Theorem 5.4]:

Theorem 2.3.2. Let $\mathcal{B}$ and $\mathcal{B}'$ be pointed pivotal bicategories and $F\colon \mathcal{B} \Rightarrow \mathcal{B}'$ be a rigid separable Frobenius functor. There is a canonical vertical transformation

$$F_*\colon \quad \mathbb{SN}_{\mathcal{B}} \Longrightarrow \mathbb{SN}_{\mathcal{B}'} \,. \tag{18}$$

from which we conclude [FSY26, Theorem 5.3]:

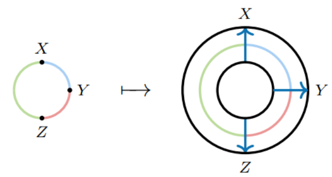
Corollary 2.3.3. Let $\mathcal{C}$ be a K-linear pivotal tensor category. There is a canonical monoidal vertical transformation

$$\mathbb{U}\mathrm{Cor}_{\mathcal{C}}\colon \quad \mathbb{SN}_{\mathcal{F}r(\mathcal{C})} \Longrightarrow \mathbb{SN}_{\mathcal{C}} \,. \tag{19}$$

The components of $\mathbb{U}\mathrm{Cor}_{\mathcal{C}}$ at an object ℓ of $\mathbb{OCB}\mathrm{ord}_2^{\mathrm{or}}$, i.e. a one-manifold ℓ is given are given by the field functor

$$\mathrm{F}_\ell : \mathbb{SN}_{\mathcal{F}r(\mathcal{C})}(\ell) \to \mathbb{SN}_{\mathcal{C}}(\ell)$$

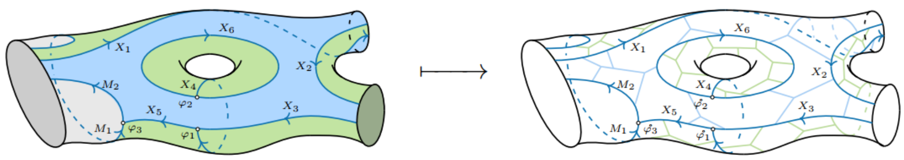
In the example of $\mathcal{F}r(\mathcal{C})$-coloured circle, we have



where the right hand side is an idempotent in $\mathrm{Cyl}^\circ(\mathcal{C}, \mathbb{S}^1)$ and thus an object in the Karoubi completion of $\mathrm{Cyl}^\circ(\mathcal{C}, \mathbb{S}^1)$. At the horizontal 1-morphism $\Sigma : a \twoheadrightarrow a'$ we need by equation (11) a natural transformation

$$\mathrm{SN}_{\mathcal{B}}(\Sigma; -, -) \Rightarrow \mathrm{SN}_{\mathcal{C}}(\Sigma; \mathrm{F}_{\ell_1}-, \mathrm{F}_{\ell_2}-)$$

It components are the linear maps given by

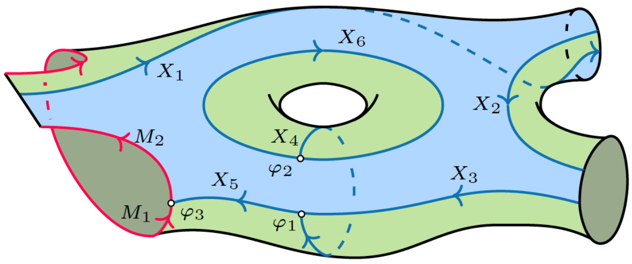


.

We can now explain the CFT-interpretation: the pivotal bicategory $\mathcal{F}r(\mathcal{C})$ has the labels suitable to decorate a surface Σ so that, given a pivotal monoidal category $\mathcal{C}$, we can assign to it a CFT correlator: the surface comes with a stratification, i.e. a two-dimensional finite CW-structure and we decorate:

- To a two-cell, we assign an object in $\mathcal{F}r(\mathcal{C})$, which describes a CFT build on the chiral data $\mathcal{C}$.
- One-cells describe line defects or boundary conditions. We assign to them bimodules, i.e. one-morphisms in $\mathcal{F}r(\mathcal{C})$, describing boundary conditions or defect conditions.
- To zero-cells, we assign morphism of bimodules which describe types of junctions.

We call such a labelled surface a worldsheet:

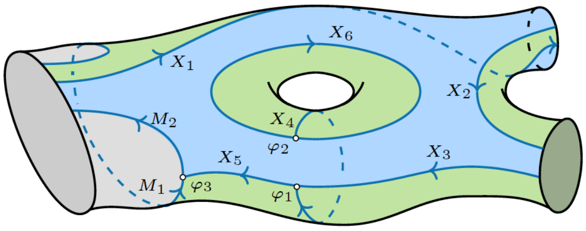


$\mathcal{S} =$

We wish to assign to it an element in a vector space provided by the modular functor $\mathbb{SN}_{\mathcal{C}}$ which carries the information on the monodromies of the conformal blocks.

Using the fact that the bicategory $\mathcal{F}r(\mathcal{C})$ is pointed, we get a bicoloring which determines a vector in the string-net space for the bicategory $\mathcal{F}r(\mathcal{C})$

$\tilde{\mathcal{S}} =$

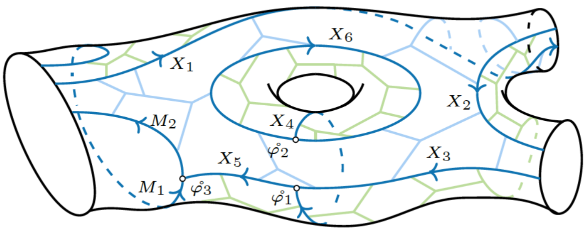


$\in \mathrm{SN}_{\mathcal{B}}(\Sigma; \ell_1, \ell_2)$

(Patches shaded in grey are labelled by the distinguished object of the pointed bicategory $\mathcal{F}r(\mathcal{C})$.)

By corollary 2.3.3,we get

$\mathrm{Cor}_{\mathcal{S}} :=$



$\in \mathrm{SN}_{\mathcal{C}}(\Sigma, \mathbb{F}\ell_1, \mathbb{F}\ell_2)$

Remarks 2.3.4. 1. These vectors obey two important consistency constraints:

- They are invariant under the action natural subgroup of the mapping class group. (The definition of a mapping class in the presence of *topological* defects is subtle, see [FSY22, (6.11)].

2. For rational conformal field theories, it has been shown that all solutions of these consistency constraints can be obtained in this way.

3. For rational conformal field theories, it can be shown that the set of the vectors $\mathrm{Cor}_{\mathcal{S}}$ for all possible worldsheets $\mathcal{S}$ span the space of conformal blocks [FSY26, Theorem 3.4].

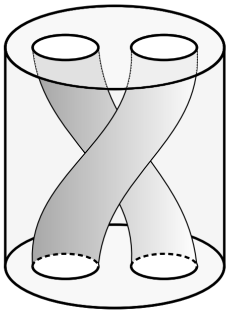
3 Lecture 3: State sum models

3.1 State sum TFT

We now discuss the construction of an extended three-dimensional topological field theory. Our goal is to extend a bicategorical monoidal functor

$$\mathrm{SN}_{\mathcal{B}} \colon \mathcal{B}\mathrm{ord}_2^{\mathrm{or,o/c}} \to \mathcal{P}\mathrm{rof}_K$$

to a symmetric monoidal functor defined on a bicategory which has all three-manifolds with corners as 2-morphisms, e.g.



Our input is a spherical fusion category $\mathcal{C}$ over the field $\mathbb{C}$ of complex numbers. Our exposition closely follows [TV17]; another exposition is [Ki11, BK10] which uses a different combinatorial geometry and works in a Poincaré dual picture.

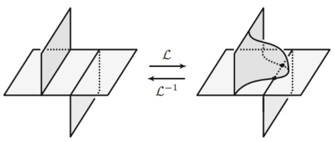
The following proposition will be used:

Proposition 3.1.1. [ENO05, Theorem 2.3]
If $\mathcal{C}$ is a spherical fusion category over the field $\mathbb{C}$, then the so-called global dimension of $\mathcal{C}$ is non-zero:

$$\mathcal{D}_{\mathcal{C}} := \sum_{i \in I} (\dim V_i)^2 \neq 0 \ .$$

Observation 3.1.2.

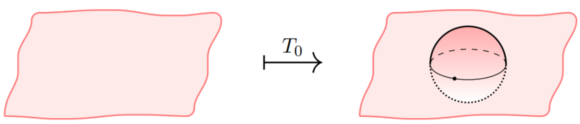
1. All manifolds are compact, oriented and piecewise linear. Following [TV17], we fix as a combinatorial datum a skeleton Δ. Locally, it looks as follows:



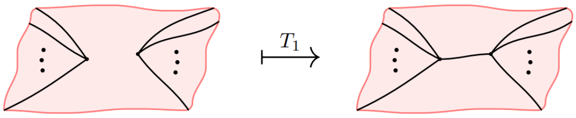
Triangulations give specific examples of skeleta. We call a manifold M with a skeleton Δ a combinatorial manifold.

2. We introduce local primary moves which relate different combinatorial 3-manifolds:

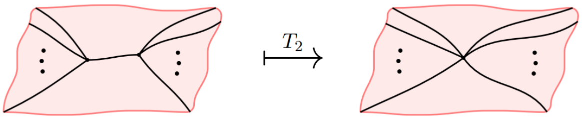
   T_0 : The *bubble move*. This move consists of first removing an open disk from a region and afterwards gluing a sphere along its equator to the boundary that results from the removal of the disk; in addition, a new edge along the equator of the sphere as well as a new vertex on that edge are added:



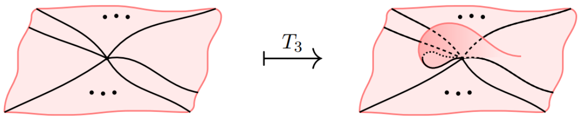
T_1 : The *phantom edge move*. This move adds a new edge by gluing its end points to two existing distinct vertices of S:



T_2 : The *contraction move*. This collapses an edge in S with distinct end points to a single vertex:



T_3 : The *percolation move*. This move pushes a branch across a vertex v in S. In more detail, an open disk whose boundary contains v is removed from a branch, and then the boundary resulting from the disk removal is glued on another branch at v:



3. By [TV17, Thm. 11.1], any two skeletons of a closed 3-manifold M can be related by a finite sequence of primary moves. This is a difficult fact of combinatorial geometry.

Observation 3.1.3. Let $\mathcal{C}$ be a spherical fusion category over the field of complex numbers.

1. A (simple) labeling l of a combinatorial three-manifold (M, Δ) is a map that assigns to an (oriented) 2-cell f of Δ a (simple) object of $\mathcal{C}$ such that $l(\overline{f}) = l(f)^*$ for the 2-cell $\overline{f}$ with opposite orientation.

2. A pair, consisting of an edge and an adjacent vertex, is called a half-edge. Any edgerelates two vertices and thus gives rise to two half-edges.

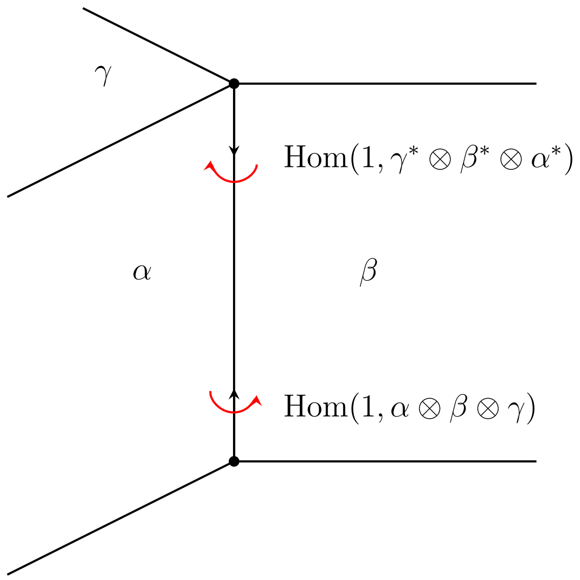
   The vector space assigned to the half-edge (e, v) given a specific labelling is a space of cyclically invariant tensors, cf. the proof of proposition 2.1.4

$$H(e, v, l) = \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, l(f_1) \otimes \ldots \otimes l(f_n))$$

3. To the other half-edge (e, v') belonging to the same edge e, we assign a vector space

$$H(e, v', l) = \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, l(f_n)^* \otimes \ldots \otimes l(f_1)^*)$$

   which is canonically in duality with $H(e, v, l)$.

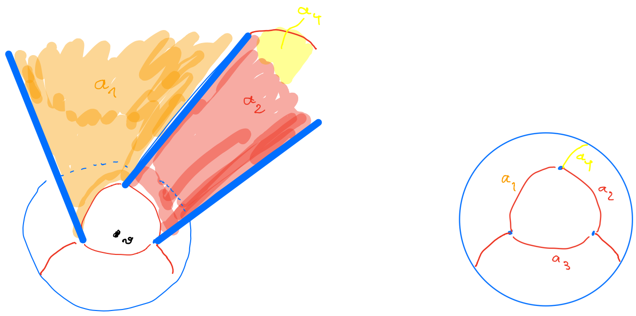


4. By standard linear algebra, there is a canonical non-zero vector in $V \otimes V^* \cong \mathrm{End}(V)$. Tensoring over all half edges leads for each labelling l to a finite-dimensional vector space

$$V(\Delta, l) := \otimes_{e \in \Delta_1} H(e, v, l) \otimes H(e, v', l)$$

with a canonical non-zero vector $\tilde{v}_{\Delta,l} \in V(\Delta, l)$.

5. On the other hand, we look at the situation from the point of view of vertices: drawing a (small) sphere around each vertex, we get a $\mathcal{C}$-labelled graph on the sphere. The $\mathcal{C}$-labelled 2-cells of the skeleton in M give lines on the sphere, labelled by objects in $\mathcal{C}$. The half-edges of the skeleton give vertices on the sphere labelled by spaces of invariant tensors.



For each vertex v, the graphical calculus on the sphere gives us a linear map

$$\mathrm{ev}_{v,l} : \bigotimes_{e \text{ incident to } v} H(e, v, l) \to \mathrm{End}_{\mathcal{C}}(1) \cong K$$

6. Consider a closed three-manifold M. The tensor product over all vertices gives a linear functional

$$\mathrm{ev}_{\Delta,l} := \bigotimes_{v \in \Delta_0} \mathrm{ev}_{v,l} : \bigotimes_{v \in \Delta_0} \bigotimes_{e \text{ incident to } v} H(e, v, l) = \bigotimes_{e \in \Delta_1} H(e, v, l) \otimes H(e, v', l) = V(\Delta, l) \to K$$

For a closed three-manifold M, we get for any choice of skeleton Δ and any choice of labelling l by evaluation on the canonical non-zero vector a number

$$Z(M, \Delta, l) := \mathrm{ev}_{\Delta,l}(\tilde{v}_{\Delta,l}) \in K \ .$$

7. One now has to sum over all possible simple labellings of the skeleton Δ with an appropriate weight. Suppose that the 2-cell f has Euler characteristic χ_f and is labelled by $l(f)$; then set

$$\omega_l := \prod_{f\in\Delta_2} \dim(U_{l(f)})^{\chi_f} \in K \ .$$

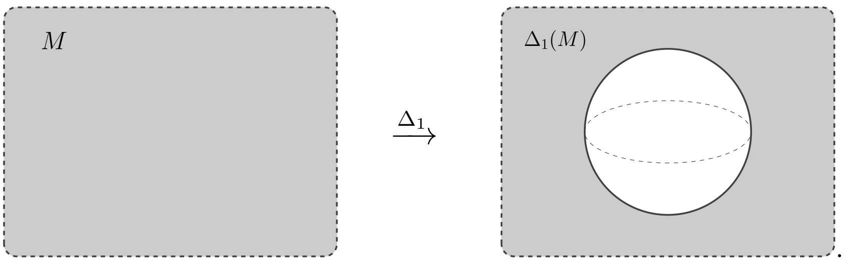
The (finite) sum over all simple labelings

$$Z_{TV}(\mathcal{C}; M, \Delta) := (\dim \mathcal{C})^{-n_3} \sum_l \omega_l Z(M, \Delta, l) \tag{20}$$

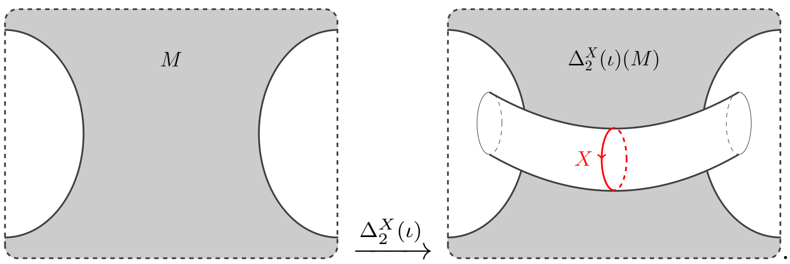
with n_3 the number of 3-cells, can be shown to invariant under the moves $T_0 - T_3$ in observation 3.1.2 and is therefore independent of the skeleton Δ. (The sum over simple labelings is at the origin of the name state-sum model.) We can thus also suppress Δ in the notation and write $Z_{TV}(\mathcal{C}; M)$.

This number will be the invariant that a three-dimensional topological field theory assigns to M: we see M as a cobordim $\emptyset \xrightarrow{M} \emptyset$ and notice that for any topological field theory $Z_{TV}(\emptyset) = K$ and so that $Z_{TV}(\mathcal{C}; M)$ is a linear map of one-dimensional vector spaces which is just the number $Z_{TV}(\mathcal{C}, M)$.

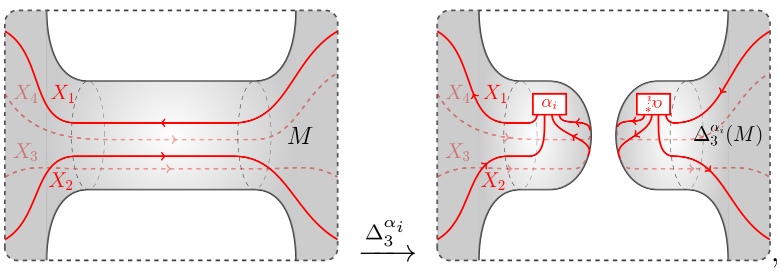
Remarks 3.1.4. 1. The Turaev-Viro invariant (20) can be explained as follows [SS25]. Assume for simplicity that all 3-cells of a combinatorial 3-manifold Δ are balls and that we can place an empty ball with some physical boundary inside at the price of a constant $(\dim \mathcal{C})^{-1}$:



Assume that we can drill through 2-cells to connect neighboring empty balls, the price of adding a boundary Wilson line labelled by the canonical coulour $X = \sum_i (\dim S_i) S_i$



Finally, assume that we can cut the solid cylinder surrounding an edge



We then get a collection of graphs on spherical balls which we evaluate by the graphical calculus.

2. State-sum formulae exist also for four-dimensional topological field theories [DR18].

To define a topological field theory Z_{TV}, we now turn to the case that the three-manifold M has a (not neceesarily connected) boundary. We have to assign vector spaces to boundaries. (To distinguish them from boundaries that appear later, we sometimes call them gluing boundaries, because they are glued in the composition of bordisms.)

Observation 3.1.5.

1. Assume that the three-manifold M with skeleton Δ has a boundary. We assume that this restricts on the gluing boundary surface Σ to the structure of a two-dimensional CW complex (Σ, δ).

2. A simple labeling l of (M, Δ) gives a labeling $\tilde{l}$ of the the edges of the two-dimensonal CW-complex δ by simple objects of $\mathcal{C}$. Hence we study now two-dimensional CW-complexes on Σ with a simple edge-labeling $\tilde{l}$ and forget about M.

3. Given a simple labelling $\tilde{l}$ of the edges of δ, we again assign to a vertex $v \in \delta_0$ the space of invariant tensors, as in Observation 3.1.3.2.

   We assign to $(\Sigma, \delta, \tilde{l})$ the vector space

$$\mathcal{H}(\Sigma, \delta, \tilde{l}) := \bigotimes_{v \in \delta} H(v, \tilde{l})$$

   and to (Σ, δ) the direct sum

$$\mathcal{H}(\Sigma, \delta) := \bigoplus_{\tilde{l}} \mathcal{H}(\Sigma, \delta, \tilde{l}) \ .$$

   - This vector space depends on the choice of skeleton δ of Σ and is therefore *not* the vector space assigned to Σ by the topological field theory we want to construct.
   - The assignment is tensorial: for a disjoint union $\Sigma_1 \sqcup \Sigma_2$ of 2-manifolds with skeleton $\delta_1 \sqcup \delta_2$, we obtain the vector space

$$\mathcal{H}(\Sigma_1 \sqcup \Sigma_2, \delta_1 \sqcup \delta_2) = \mathcal{H}(\Sigma_1, \delta_1) \otimes \mathcal{H}(\Sigma_2, \delta_2) \ .$$

   - Upon change of orientation, we obtain the dual vector space

$$\mathcal{H}(\overline{\Sigma}, \delta) \cong \mathcal{H}(\Sigma, \delta)^* \ .$$

4. To compare to the Kitaev model, we introduce the following sum of commuting idempotents on $V(\delta)$

$$\mathsf{H}_{vertex} := \sum_{v \in \delta_1} (\mathrm{id} - A_v) \ .$$

   and define

$$\hat{V}(H, \Sigma, \delta) = \{v \in H^{\otimes n} : \mathsf{H}_{vertex} v = 0\} \subset V(\delta)$$

   On the edges, we had in the Kitaev model H as a bimodule which, for H semisimple decomposes into a direct sum of bimodules

$$H \cong \oplus_{s \in S} S_i \boxtimes S_i^*$$

where the sum is over isomorphism classes of simple objects. This provides an isomorphism of vector spaces

$$\Phi : \quad \hat{V}(H;\Sigma,\delta) \longrightarrow \mathcal{H}(H\text{−mod};\Sigma,\delta) \ . \tag{21}$$

For more details, we refer to [BK12, Lemma 4.2].

We now assume that the surface Σ is the boundary of M.

Observation 3.1.6.

1. We now assume that Σ is the gluing boundary of a three-manifold M with a skeleton Δ and that the structure of a combinatorial two-manifold δ on Σ is obtained by restriction from Δ. We fix a labelling of Δ by simple objects in $\mathcal{C}$; here, the 2-cells contained in the (gluing) boundary Σ are not labelled. Evaluations are only applied to vertices in the interior of M. Edges of Δ ending on Σ will be called "dangling edges". We now obtain an evaluation:

$$\mathrm{ev}_{\Delta,l} := \otimes_{v\in\Delta_0^{\mathrm{int}}}\mathrm{ev}_{v,l} : \quad V_{\Delta,l} \to \otimes_{e \text{ dangling}} V_e = \mathcal{H}(\Sigma,\delta,\tilde{l}) \subset \mathcal{H}(\Sigma,\delta)$$

2. One performs a sum over labellings with the same weights as in Observation 3.1.3.7 and obtains a vector $v_\Delta \in H(\Sigma,\delta)$. Using the moves in observation 3.1.2, one shows that this vector is independent on the part of the skeleton in the interior of M.

3. For a combinatorial 2-manifold (Σ,δ), we get the linear maps associated to the cylinders

$$A_{\Sigma,\delta} := H(\Sigma\times[0,1]) : \quad \mathcal{H}(\Sigma,\delta) \to \mathcal{H}(\Sigma,\delta) \ ,$$

   which do not depend on the details of Δ in the interior of the cylinder. The composition of two cylinders is again a cylinder. Thus, as a consequence of 2., the maps are idempotents: $A^2_{\Sigma,\delta} = A_{N\Sigma\delta}$. If we already had a topological field theory, this should be the identity.

   To a combinatorial 2-manifold (Σ,δ), we therefore assign the vector space

$$Z_{TV}(\mathcal{C};\Sigma,\delta) := \mathrm{Im}\,(A_{\Sigma,\delta}) \subset \mathcal{H}(\Sigma,\delta) \ .$$

   Notice that we have again realized a relative situation of a vector space depending on a discrete geometry and a subspace.

4. One can now show [BK12, Lemma 4.3] that the following diagram involving the isomorphism Φ from (21) and

$$\mathsf{H}_{plaqu} := \sum_p (\mathrm{id} - B_p)$$

   commutes:

$$\begin{array}{ccc} \hat{V}(\Sigma,\delta) & \xrightarrow{\ \Phi\ } & \mathcal{H}(\Sigma,\delta) \\ {\scriptstyle P_{plaqu}}\big\downarrow & & \big\downarrow{\scriptstyle A_{\Sigma,\delta}} \\ \hat{V}(\Sigma,\delta) & \xrightarrow{\ \Phi\ } & \mathcal{H}(\Sigma,\delta) \end{array}$$

   As a consequence, we can identify $Z_{TV}(H\text{−mod};\Sigma,\delta)$ and $Z_K(H;\Sigma,\delta)$

5. Given two different combinatorial structures δ, δ' on Σ, we can find a skeleton on the cylinder $\Sigma \times I$ which restricts on the boundary of the cylinder to these two combinatorial structures. This gives us linear maps $\mathcal{H}(\Sigma, \delta) \to \mathcal{H}(\Sigma, \delta')$ which are independent on the choice of skeleton and restrict to an isomorphism on the subspace $Z_{TV}(\Sigma, \delta)$. The vector space $Z_{TV}(\mathcal{C}; \Sigma_1)$ is obtained by identifying all these vector spaces using the linear maps provided by cylinders. As a consequence of the isomorphism induced by Φ, also the Kitaev ground state spaces do not depend on δ.

6. For a cobordism $\Sigma_1 \xrightarrow{M} \Sigma_2$, we denote by $Z_{TV}(M)$ the restriction of the linear map $H(M, \Delta)$ to $Z_{TV}(\Sigma)$. This defines a three-dimensional topological field theory Z_{TV} : $\mathrm{Cob}(3,2) \to \mathrm{vect}(K)$. In particular, we get representations of mapping class groups and all the structure to get quantum gates.

For the proof of all these statements, we refer to the book [TV17]. In our construction, we have assigned objects of a spherical fusion category to 2-cells; no braiding on this category is required. In the same book, also the extension of the theory to circles labelled by objects in the Drinfeld center is described.

We can also show [Ki11, Theorem 5.1]:

Theorem 3.1.7.
Let $\mathcal{C}$ a spherical fusion category. The vector spaces assigned to a closed oriented surface Σ by the string-net construction and the Turaev-Viro construction are canonically isomorphic.

Remarks 3.1.8.

1. One would like to lift this to an isomorphism of modular functors: for a spherical fusion category $\mathcal{C}$, the string-net construction, the Turaev-Viro construction and, if $\mathcal{C} = H$−mod, the Kitaev construction give rise to the same modular functors. I am not aware of a fully worked out proof, including the representations of the mapping class groups.

2. It is also known [CGPT20, Theorem 2.6] that the Kuperberg invariant for a Hopf algebra H equals, up to a constant factor, the Turaev-Viro invariant:

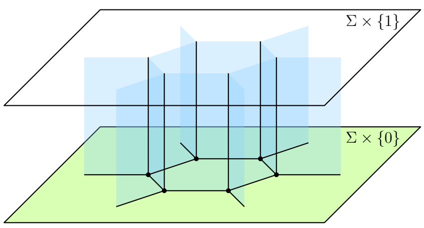
$$\frac{1}{\dim H} Z_{\mathrm{Kup}}(H, M) = Z_{TV}(H\text{−mod}, M) \ .$$

For a finite group G, the Dijkgraaf-Witten invariant coincides with the Turaev-Viro invariant of the group algebra $K[G]$ with its canonical pivotal structure.

3.2 Tensor networks

Tensor networks are a popular tool in the theory of quantum computing. Following [LFHSV], we show how they are related to the Turaev-Viro construction. Consider again (Σ, δ), a closed oriented surface with a two-dimensional CW complex.

We extend it using the three-manifold $M_\Sigma := \Sigma \times [0,1]$ with a cell decomposition Δ obtained by crossing δ with intervals,



.

The three manifold M_Σ has two boundary components.

- We consider $\Sigma \times \{1\}$ with δ and the vector space $\mathcal{H}(\Sigma, \delta)$. It will be convenient to consider for each vertex $v \in \delta$ the vector space of invariant tensors

$$\mathcal{H}_{phys,v} : \bigoplus_{\tilde{l}_v} H(v, \tilde{l})$$

where the direct sum is over simple labelings of the edges adjacent to v. This space only depends on the number of edges incident to v. It is called the physical space at the vertex v. (If we work, for example, with a hexagonal lattice on a torus $\mathbb{T}^2$, then all vector spaces are the same.)

Then we have a canonical embedding

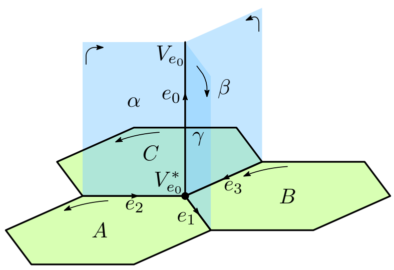
$$\mathcal{H}(\Sigma, \delta) \hookrightarrow \otimes_{v \in \delta} \mathcal{H}_{phys,v}$$

- The other boundary $\Sigma \times \{0\}$ is a physical bounday. This means that we pick a semisimple $\mathcal{C}$-module category $\mathcal{M}$ and use simple objects in $\mathcal{M}$ for the 2-cells contained in $\Sigma \times \{0\}$. (We can add boundary Wilson lines, labeled by module functors, i.e. objects in $\mathrm{End}_{\mathcal{C}}(\mathcal{M})$ or even Wilson networks.)

By the same construction as described above, considering M_Σ as a cobordism $M_\Sigma : \emptyset \to \Sigma \times \{1\}$, we get a linear map

$$Z_{TV}(M_\Sigma) : \quad K \to \mathcal{H}(\Sigma, \delta) \hookrightarrow \otimes_{v \in \delta} \mathcal{H}_{phys,v}$$

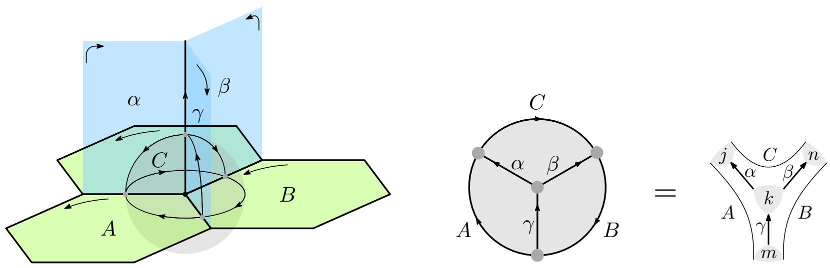
We spell out the state-sum variables:



- $\alpha \in \mathcal{C}$ in monoidal category to (blue) plaquettes in interior
- $A \in \mathcal{M}$ in a $\mathcal{C}$-module category $\mathcal{M}$ to (green) plaquette on the physical boundary
- No state-sum variables on the gluing boundary $\Sigma \times \{1\}$

The invariant tensors assigned to half-edges are now constructed from Hom-spaces of the monoidal category $\mathcal{C}$ for half-edges in the interior and Hom spaces in $\mathcal{M}$ for edges on the physical boundary $\Sigma \times \{0\}$.

We now study the evaluation at the vertices, using graphical calculus on $\mathbb{S}^2$:



For hexagonal lattices on Σ, we get tetrahedra on $\mathbb{S}^2$ and thus 6j-symbols for the monoidal category and the module category. This is called a PEPS tensor (projected entangled pair state).

Remarks 3.2.1.
1. We thus get states in $\otimes_{v\in\delta}\mathcal{H}_{phys,v}$, interpreted as a space quantum mechanical many particle states, by contracting PEPS tensors. These states turn out to have special properties, e.g. entropy related to lines rather than surfaces.
2. By including boundary Wilson lines on $\Sigma \times \{0\}$, we can get more states and encode symmetries of the system in terms of tensors again obtained by evaluations of graphs on $\mathbb{S}^2$.

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