Symmetric tensor categories

Pavel Etingof (MIT)

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Lecture 1

The model example: $\mathrm{Rep}_{\mathbb{k}}(G)$

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We would now like to axiomatize the categorical structures visible in $\mathrm{Rep}_{\Bbbk}(G)$ without referring to underlying vector spaces.

Monoidal category: data

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called the **associativity constraint**;

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with the following properties.

Monoidal category: pentagon axiom

First, the associativity constraint is required to satisfy the **pentagon axiom**: for all $W, X, Y, Z \in \mathcal{C}$, the diagram

$$\begin{tikzcd}
((W \otimes X) \otimes Y) \otimes Z \arrow[r, "a_{W \otimes X, Y, Z}"] \arrow[d, "a_{W,X,Y} \otimes \mathrm{id}_Z"'] & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y \otimes Z}"] \\
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W \otimes ((X \otimes Y) \otimes Z) \arrow[ru, "\mathrm{id}_W \otimes a_{X,Y,Z}"'] &
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commutes. Equivalently,

$$a_{W,X,Y \otimes Z}\, a_{W \otimes X,Y,Z} = (\mathrm{id}_W \otimes a_{X,Y,Z})\, a_{W,X \otimes Y,Z}\, (a_{W,X,Y} \otimes \mathrm{id}_Z).$$

Monoidal category: unit axiom

Next, the unit object $\mathbf{1}$ is required to satisfy

$$\mathbf{1} \otimes \mathbf{1} \cong \mathbf{1},$$

and tensor multiplication by $\mathbf{1}$ on either side must be an equivalence:

$$L_{\mathbf{1}} : \mathcal{C} \to \mathcal{C}, \qquad X \mapsto \mathbf{1} \otimes X,$$

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The **Maclane's coherence theorem** implies that we may ignore parentheses and unit objects inside tensor products.

Braided monoidal category: data

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They imply that for every object $X \in \mathcal{C}$ the object $X^{\otimes n}$ carries a natural action of the **braid group** B_n, with generator b_i acting by c in the i-th and $i+1$-th factors.

Braided monoidal category: hexagon axioms

1. For all $X, Y, Z \in \mathcal{C}$, the diagram

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X \otimes (Z \otimes Y) \arrow[r, "a^{-1}_{X,Z,Y}"'] & (X \otimes Z) \otimes Y \arrow[r, "c_{X,Z} \otimes \mathrm{id}_Y"'] & (Z \otimes X) \otimes Y \arrow[u, "a_{Z,X,Y}"']
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Symmetric monoidal category

A braided monoidal category is **symmetric** if the braiding is involutive:

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In $\mathrm{Rep}_{\Bbbk}(G)$, c_{VW} is the flip

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Rigidity: dual data

Let $\mathcal{C}$ be monoidal. A **left dual** of $X \in \mathcal{C}$ is an object X^* with morphisms

$$\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}, \qquad \mathrm{coev}_X : \mathbf{1} \to X \otimes X^*,$$

satisfying the two triangular identities.

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A **right dual** is defined analogously, with evaluation $X \otimes {}^*X \to \mathbf{1}$ and coevaluation $\mathbf{1} \to {}^*X \otimes X$. A monoidal category is **rigid** if every object has both a left and a right dual.

Rigidity: triangular identities

The left-dual triangular identities are the commutativity of the following two diagrams:

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X \arrow[r, "\mathrm{coev}_X \otimes \mathrm{id}_X"] \arrow[rrr, bend right, "\mathrm{id}_X"'] & (X \otimes X^*) \otimes X \arrow[r, "a_{X,X^*,X}"] & X \otimes (X^* \otimes X) \arrow[r, "\mathrm{id}_X \otimes \mathrm{ev}_X"] & X
\end{tikzcd}$$

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X^* \arrow[r, "\mathrm{id}_{X^*} \otimes \mathrm{coev}_X"] \arrow[rrr, bend right, "\mathrm{id}_{X^*}"'] & X^* \otimes (X \otimes X^*) \arrow[r, "a^{-1}_{X^*,X,X^*}"] & (X^* \otimes X) \otimes X^* \arrow[r, "\mathrm{ev}_X \otimes \mathrm{id}_{X^*}"] & X^*
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Equivalently, the corresponding composites are id_X and id_{X^*}.

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Note that if X^* or ${}^*X$ exists for a given X then it is unique up to a unique isomorphism, and that in a symmetric category we have a canonical isomorphism ${}^*X \cong X^*$, so there is only one dual.

Rigidity in $\mathrm{Rep}_{\Bbbk}(G)$

For $V \in \mathrm{Rep}_{\Bbbk}(G)$, set

$$V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk),$$

with contragredient G-action. Evaluation and coevaluation are

$$\mathrm{ev} : V^* \otimes V \to \Bbbk, \qquad \varphi \otimes v \mapsto \varphi(v),$$

$$\mathrm{coev} : \Bbbk \to V \otimes V^*, \qquad 1 \mapsto \sum_i v_i \otimes v_i^*,$$

for a basis $\{v_i\}$ and its dual basis $\{v_i^*\}$.

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for a basis $\{v_i\}$ and its dual basis $\{v_i^*\}$. These maps are G-equivariant and satisfy the triangular identities.

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- the bifunctor $\otimes$ is $\Bbbk$-bilinear.

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Following the book *Tensor Categories* by P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, a **tensor category** over $\Bbbk$ is an essentially small abelian $\Bbbk$-linear rigid monoidal category such that:

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A braided tensor category is called **symmetric** is its braiding is symmetric.

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It is a whole world where we can do algebra and representation theory!

Examples of symmetric tensor categories

For example, $\mathrm{Rep}_{\mathbb{k}}(G)$ is a symmetric tensor category. The most basic case is $G = 1$, which gives **the category of finite dimensional $\mathbb{k}$-vector spaces**, $\mathrm{Vec}_{\mathbb{k}}$.

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A **rational representation** of an affine group scheme G is a finite-dimensional $\mathcal{O}(G)$-**comodule** V, i.e., a vector space V with a **coaction**

$$\rho : V \to V \otimes \mathcal{O}(G)$$

which defines an action of the algebra $\mathcal{O}(G)^*$ on V. It is easy to see that rational representations of an affine group scheme G form a symmetric tensor category $\mathrm{Rep}(G)$.

Tensor functors

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Let $\mathcal{C}, \mathcal{D}$ be monoidal categories. A **monoidal functor** $F : \mathcal{C} \to \mathcal{D}$ consists of a functor and coherent isomorphisms

$$J_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y), \qquad \nu : \mathbf{1}_{\mathcal{D}} \xrightarrow{\sim} F(\mathbf{1}_{\mathcal{C}}),$$

compatible with the associativity and unit data. A **braided (symmetric) monoidal functor** is a monoidal functor between braided (symmetric) monoidal categories compatible with the braidings.

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If $\mathcal{C}, \mathcal{D}$ are tensor categories then a monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is called a **tensor functor** if it is $\Bbbk$-linear and faithful (hence exact). A **fiber functor** is a symmetric tensor functor to the target $\mathrm{Vec}_{\Bbbk}$.

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Thus we would like to explain why the previous examples **are equivalent** to $\mathrm{Rep}(G)$ for some affine group scheme G as symmetric tensor categories.

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This is a **fiber functor**: a symmetric tensor functor to $\mathrm{Vec}_{\Bbbk}$.

Tensor automorphisms of a fiber functor

Let $\omega : \mathcal{C} \to \mathrm{Vec}_{\Bbbk}$ be a fiber functor. Define the affine group functor

$$\underline{\mathrm{Aut}}_{\otimes}(\omega) : R \mapsto \mathrm{Aut}^{\otimes}(\omega_R), \qquad \omega_R(X) = \omega(X) \otimes_{\Bbbk} R.$$

Thus an R-point of $\underline{\mathrm{Aut}}_{\otimes}(\omega)$ is a family

$$\{g_X : \omega(X) \otimes R \xrightarrow{\sim} \omega(X) \otimes R\}_{X \in \mathcal{C}}$$

which is natural in X, preserves tensor products and the unit, and is invertible.

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For $\mathcal{C} = \mathrm{Rep}(G)$ and $\omega = \omega_G$, evaluation of representations gives

$$G \cong \underline{\mathrm{Aut}}_{\otimes}(\omega_G).$$

The Coend of matrix coefficients

The functor $\underline{\mathrm{Aut}}_{\otimes}(\omega)$ is represented by a Hopf algebra constructed from matrix coefficients. Set

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Concretely, this is the quotient of

$$\bigoplus_{X \in \mathcal{C}} \omega(X)^* \otimes \omega(X)$$

by the relations

$$\omega(f)^* \varphi \otimes v - \varphi \otimes \omega(f) v \in \omega(Y)^* \otimes \omega(Y)$$

for all morphisms $f : X \to Y$, where $\varphi \in \omega(Y)^*$ and $v \in \omega(X)$.

Hopf algebra structure on the Coend

Write the class of $\psi \otimes v \in \omega(X)^* \otimes \omega(X)$ as $[\psi \mid v]_X$. The coalgebra structure is the usual one for matrix coefficients:

$$\Delta([\psi \mid v]_X) = \sum_i [\psi \mid e_i]_X \otimes [e_i^* \mid v]_X, \qquad \varepsilon([\psi \mid v]_X) = \psi(v),$$

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The tensor structure of ω defines multiplication:

$$[\psi \mid v]_X \, [\varphi \mid w]_Y = [\psi \otimes \varphi \mid v \otimes w]_{X \otimes Y},$$

after identifying $\omega(X \otimes Y) \cong \omega(X) \otimes \omega(Y)$. Duals give the antipode.

Tannakian reconstruction

Let $\mathcal{C}$ be a symmetric tensor category and let

$$\omega : \mathcal{C} \to \mathrm{Vec}_{\mathbb{k}}$$

be a fiber functor. Put

$$G := \underline{\mathrm{Aut}}_{\otimes}(\omega), \qquad \mathcal{O}(G) = \int^{X \in \mathcal{C}} \omega(X)^* \otimes \omega(X).$$

Then the canonical coactions on $\omega(X)$ define a symmetric tensor equivalence

$$\mathcal{C} \simeq \mathrm{Rep}(G).$$

This shows that all our examples are of the form $\mathrm{Rep}(G)$ for an affine group scheme G.

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It was shown by Deligne and Milne that the fiber functor is unique up to isomorphism; consequently G is uniquely determined up to conjugation.

Abstract groups and proalgebraic completion

Let us now explain how this works in our first example. Let Γ be an abstract group. The forgetful functor gives

$$\widehat{\Gamma} := \underline{\mathrm{Aut}}_{\otimes}(\omega_\Gamma), \qquad \mathrm{Rep}_{\Bbbk}(\Gamma) \simeq \mathrm{Rep}(\widehat{\Gamma}).$$

The affine group scheme $\widehat{\Gamma}$ is the **proalgebraic completion** of Γ over $\Bbbk$.

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Its coordinate algebra is the subalgebra of $\mathrm{Fun}(\Gamma, \Bbbk)$ spanned by the matrix coefficients

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Thus the group scheme $\widehat{\Gamma}$ is reduced, i.e., it is a proalgebraic group. Namely, it is the inverse limit of algebraic groups G equipped with a homomorphism from Γ, i.e.,

$$\widehat{\Gamma} = \varprojlim_{\Gamma \to G} G.$$

In particular, $\widehat{\Gamma}$ carries a canonical (universal) homomorphism $\Gamma \to \widehat{\Gamma}$ (which may or may not be injective).

Tannakian categories

A symmetric tensor category $\mathcal{C}$ over $\mathbb{k}$ is called **Tannakian** if it admits a fiber functor

$$\omega : \mathcal{C} \to \mathrm{Vec}_{\mathbb{k}} \,.$$

Equivalently, by reconstruction,

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So it is natural to ask: **are there other examples of symmetric tensor categories**?

The first non-Tannakian example - supervector spaces

The answer turns out to be **yes**. Assume $\mathrm{char}\,\Bbbk \neq 2$. Let $\mathrm{sVec}_{\Bbbk}$ be the category of finite-dimensional **supervector spaces**

$$V = V_{\bar{0}} \oplus V_{\bar{1}}.$$

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The underlying tensor category is equivalent to $\mathrm{Rep}_{\Bbbk}(\mathbb{Z}/2)$, but the symmetric structure is different.

But why is it a genuinely new example?

Trace and categorical dimension

Let X be an object of a symmetric tensor category and let $f : X \to X$. The **categorical trace** is

$$\mathrm{Tr}(f) : \mathbf{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^* \xrightarrow{f \otimes 1} X \otimes X^* \xrightarrow{c_{X,X^*}} X^* \otimes X \xrightarrow{\mathrm{ev}_X} \mathbf{1}.$$

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In $\mathrm{Vec}_\Bbbk$, the trace is the usual trace and dimension is the usual dimension viewed as an element of $\Bbbk$. In $\mathrm{sVec}_\Bbbk$, the trace is the **supertrace** $\mathrm{Tr}(f) = \mathrm{Tr}(f_0) - \mathrm{Tr}(f_1)$, and the dimension is the **superdimension**

$$\dim(V) = \dim_\Bbbk V_{\bar{0}} - \dim_\Bbbk V_{\bar{1}}.$$

Invertible objects distinguish sVec from Tannakian categories

An object X in a monoidal category is **invertible** if there exists Y such that

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Equivalently, in a rigid category, $X \otimes X^* \cong \mathbf{1}$.

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How can we generalize this example?

Affine supergroup schemes

Affine supergroup schemes

An **affine superscheme** X over $\Bbbk$ is represented by a commutative algebra $\mathcal{O}(X) = \mathcal{O}(X)_{\bar{0}} \oplus \mathcal{O}(X)_{\bar{1}}$ in $\mathrm{sVec}_{\Bbbk}$ (a.k.a. **supercommutative algebra**).

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Equivalently, G is a representable group-valued functor of points on supercommutative $\Bbbk$-algebras.

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Basic examples of affine supergroup schemes are classical algebraic supergroups, such as $GL(m|n)$ and $Osp(m|2n)$.

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Likewise, an **affine supergroup scheme** G over $\Bbbk$ is represented by a commutative Hopf algebra $\mathcal{O}(G) = \mathcal{O}(G)_{\bar{0}} \oplus \mathcal{O}(G)_{\bar{1}}$ in $\mathrm{sVec}_{\Bbbk}$ (a.k.a. **supercommutative Hopf superalgebra**).

Equivalently, G is a representable group-valued functor of points on supercommutative $\Bbbk$-algebras.

Basic examples of affine supergroup schemes are classical algebraic supergroups, such as $GL(m|n)$ and $Osp(m|2n)$.

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There is a slight generalization of this example. Suppose $z \in G(\Bbbk)$ is an element of order 2 acting on $\mathcal{O}(G)$ by parity. Then we may consider the category $\mathcal{C} \simeq \mathrm{Rep}(G, z)$ of representations of G on which z acts by parity. This is also a symmetric tensor category, and $\mathrm{Rep}(G) \cong \mathrm{Rep}(\{1, z\} \ltimes G, z)$, so it generalizes the previous example.

Superfiber functors and super-Tannakian categories

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Super-Tannakian reconstruction

Let

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be a superfiber functor. Deligne showed that in this case $\mathcal{C} = \mathrm{Rep}(G, z)$ for a unique pair (G, z) up to conjugation.

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The coordinate Hopf superalgebra is again a Coend of matrix coefficients:

$$\mathcal{O}(G) = \int^{X \in \mathcal{C}} \omega(X)^* \otimes \omega(X),$$

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Since ω is unique, so is (G, z).

Are there examples beyond super-Tannakian categories?

We have thus enlarged the Tannakian class from fiber functors to $\mathrm{Vec}_{\Bbbk}$ to fiber functors to $\mathrm{sVec}_{\Bbbk}$, i.e., to super-Tannakian categories. The next question is:

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Without restrictions on size of tensor products, the answer is **no**: the **Deligne categories** which we will consider later provide examples of non-super-Tannakian behavior.

However, the answer in characteristic zero becomes **yes** under the condition of **moderate growth**.

Moderate growth

Let $\mathcal{C}$ be a tensor category. For $X \in \mathcal{C}$, let

$$\ell_n(X) := \ell(X^{\otimes n})$$

be the length of the n-th tensor power. The category $\mathcal{C}$ has **moderate growth** if for every X there exists $C_X > 0$ such that

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$$\ell(X) \leq \dim_{\Bbbk} \omega(X)_{\bar{0}} + \dim_{\Bbbk} \omega(X)_{\bar{1}}.$$

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Thus every super-Tannakian category has moderate growth.

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In 2002 Deligne proved the following remarkable theorem, which is quite non-trivial.

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Let $\mathbb{k}$ *be algebraically closed of characteristic* 0*. A symmetric tensor category over* $\mathbb{k}$ *has moderate growth if and only if it is super-Tannakian.*

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Equivalently, under moderate growth there exists a superfiber functor

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This shows that while we can do algebra in any symmetric tensor category, in characteristic zero and moderate growth setting such algebra can be described in classical terms, as ordinary equivariant algebra under a supergroup scheme.

Lecture 2

Deligne categories without moderate growth

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The example is Deligne's interpolation $\mathrm{Rep}\, GL_t$ of $\mathrm{Rep}_{\mathbb{C}}\, GL_n$ in which n is replaced by a parameter $t \in \mathbb{C}$.

For $t \notin \mathbb{Z}$, the category $\mathrm{Rep}\, GL_t$ is semisimple symmetric tensor category, but it is not super-Tannakian because it does not have moderate growth.

The classical input: mixed tensor powers

To define $\operatorname{Rep} GL_t$, we first need to describe the classical category $\operatorname{Rep}_{\mathbb{C}} GL_n$ in a way that does not use that n is an integer.

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The goal is to describe the tensor subcategory generated by V and V^* using only invariant theory.

Thus one studies the spaces

$$\operatorname{Hom}_{GL_n}([r, s]_n, [p, q]_n)$$

and their behavior when n is large relative to r, s, p, q.

Stable Hom spaces

By rigidity in $\operatorname{Rep} GL_n$, there is a canonical identification

$$\operatorname{Hom}_{GL_n}([r,s]_n,[p,q]_n) \cong \operatorname{Hom}_{GL_n}(V^{\otimes r} \otimes V^{\otimes q}, V^{\otimes p} \otimes V^{\otimes s}) = \operatorname{Hom}_{GL_n}(V^{\otimes r+q}, V^{\otimes p+s}).$$

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The dependence on n is simple: every closed loop produced in a composition contributes the scalar n.

Walled Brauer diagrams: the wall rule

The stable Hom spaces are described by **walled Brauer diagrams**. A diagram for a morphism $[r, s] \to [p, q]$ has a vertical wall separating the contravariant side (r copies of V at the bottom and p at the top, left of the wall) from the covariant side (s copies of V^* at the bottom and q at the top, right of the wall).

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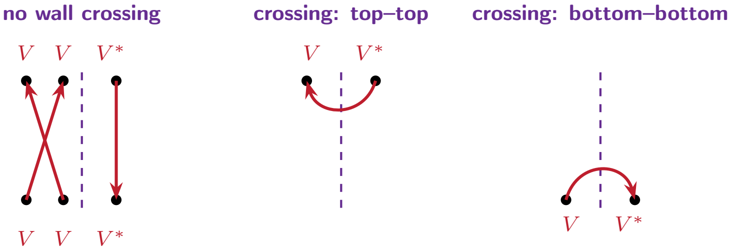
- A strand which does not cross the wall goes from the bottom row to the top row on the left and from the top row to the bottom row on the right.



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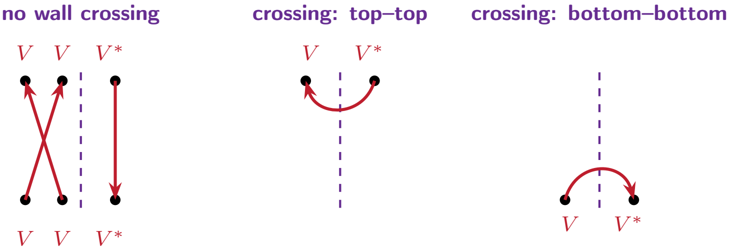
- A strand which does not cross the wall goes from the bottom row to the top row on the left and from the top row to the bottom row on the right.
- A strand which crosses the wall bends back: it connects two top endpoints (oriented to the left), or two bottom endpoints (oriented to the right).



A basis diagram

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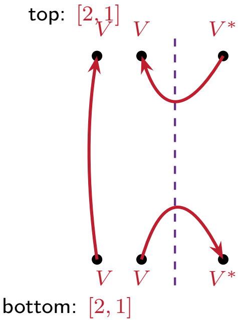
For example, the following diagram is an endomorphism of $[2,1]$. It has one ordinary bottom-to-top strand, one bottom cup crossing the wall, and one top cap crossing the wall.



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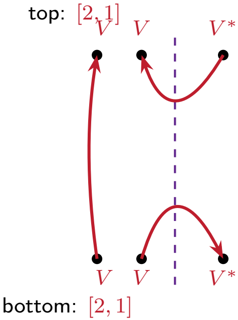

The diagram encodes a matching together with the orientation rule; there is no difference between over-crossings and under-crossings.

Composition and the loop scalar

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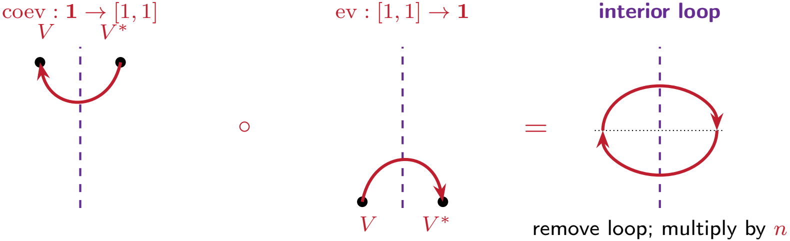
The basic example is the composition

$$\mathbf{1} \xrightarrow{\text{coev}} [1,1] \xrightarrow{\text{ev}} \mathbf{1}, \qquad \text{ev} \circ \text{coev} = n\ \text{id}_{\mathbf{1}}\,.$$



Polynomiality and interpolation

For fixed p, q, r, s and large n, the composition gives bilinear maps

$$\mathrm{Hom}([r,s],[p,q]) \times \mathrm{Hom}([p,q],[a,b]) \longrightarrow \mathrm{Hom}([r,s],[a,b]).$$

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This gives the diagrammatic interpolation of the tensor category generated by the defining representation of GL_n and its dual.

The category $\widetilde{\mathrm{Rep}}GL_t$

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The tensor product is $[p, q] \otimes [r, s] = [p + r, q + s]$ (**putting diagrams next to each other and moving the V to left and V^* of the right**) and duality is $[r, s]^* = [s, r]$ (**rotating diagrams $180°$**). The unit object is $\mathbf{1} = [0, 0]$. The braiding is swapping two diagrams which are next to each other.

Symmetric pseudotensor categories

Definition. A **symmetric pseudotensor category** over $\Bbbk$ is a Karoubian (i.e., additive idempotent complete) $\Bbbk$-linear rigid symmetric monoidal category $\mathcal{C}$ such that

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(X, Y) < \infty, \qquad \mathrm{End}_{\mathcal{C}}(\mathbf{1}) = \Bbbk.$$

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The category $\mathrm{Rep}\, GL_t := \mathrm{Kar}(\widetilde{\mathrm{Rep}} GL_t)$ (the additive idempotent completion of $\widetilde{\mathrm{Rep}} GL_t$) is naturally a symmetric pseudotensor category.

Endomorphism algebras and semisimplicity

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Lemma. If $W_{r,s}(t)$ is semisimple for all r, s, then $\mathrm{Rep}\, GL_t$ is a semisimple tensor category.

Generic rank

Theorem. If $t \notin \mathbb{Z}$, then the algebras $W_{r,s}(t)$ are semisimple for all $r, s \geq 0$.

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Example. The algebra $W_{1,1}(t)$ is 2-dimensional, with basis 1 and x (composition of coevaluation with evaluation) and multiplication defined by $x^2 = tx$. Thus $W_{1,1}(t) = \mathbb{C}[x]/(x^2 - tx)$ which is semisimple iff $t \neq 0$.

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Namely, $V_{\lambda,\mu}$ interpolates the irreducible GL_n-module with highest weight

$$(\lambda_1, \dots, \lambda_r, 0, \dots, 0, -\mu_s, \dots, -\mu_1).$$

for $n \geq r + s$.

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Thus

$$\ell(V^{\otimes d}) \geq \sqrt{d!},$$

which is faster than exponential growth.

Tensor ideals

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The **quotient** $\mathcal{C}/\mathcal{I}$ is the symmetric pseudotensor category which has the same objects as $\mathcal{C}$ and morphism spaces

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)/\mathcal{I}(X, Y).$$

Negligible morphisms

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For $f : X \to Y$ and $g : Y \to X$, the composite $gf \in \mathrm{End}(X)$ has categorical trace

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The negligible morphisms form a tensor ideal, denoted $\mathcal{N} \subseteq \mathcal{C}$.

Semisimplicity of the negligible quotient

Lemma. (Benson) Suppose that for every object $X \in \mathcal{C}$, every nilpotent endomorphism $a \in \mathrm{End}_{\mathcal{C}}(X)$ satisfies $\mathrm{Tr}(a) = 0$ (**trace zero property**).

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(ii) is straightfoirward.

Corollary If $\mathcal{C}$ has trace zero property then the quotient $\mathcal{C}/\mathcal{N}$ is a semisimple symmetric tensor category, with simples being the indecomposables of $\mathcal{C}$ of nonzero dimension.

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Consequently,

$$(\operatorname{Rep} GL_t)^{ss} \simeq \operatorname{Rep}_{\mathbb{C}} GL_n.$$

Abelian envelopes and universal property

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If $t \notin \mathbb{Z}$, write $\operatorname{Rep}^{ab} GL_t := \operatorname{Rep} GL_t$. Then for all $t \in \mathbb{C}$ $\operatorname{Rep} GL_t$ have the following **universal property**: symmetric tensor functors $H : \operatorname{Rep}^{ab} GL_t \to \mathcal{D}$ to a symmetric tensor category $\mathcal{D}$ over $\mathbb{C}$ correspond to objects $X \in \mathcal{D}$ of non-moderate growth and dimension t. Namely, $X = H([1, 0])$.

Ultrafilters

Another method of constructing interpolating categories is using ultrafilters.

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Alternatively, one may say that an ultrafilter on I is a homomorphism $\chi : \mathrm{Fun}(I, \mathbb{F}_2) \to \mathbb{F}_2$, i.e., a maximal ideal ($\mathrm{Ker}\chi$) in this ring. Then $A \in \mathcal{F}$ iff $\chi(1_A) = 1$.

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Alternatively, one may say that an ultrafilter on I is a homomorphism $\chi : \mathrm{Fun}(I, \mathbb{F}_2) \to \mathbb{F}_2$, i.e., a maximal ideal ($\mathrm{Ker}\chi$) in this ring. Then $A \in \mathcal{F}$ iff $\chi(1_A) = 1$.

There is a boring class of ultrafilters: given $x \in I$, we have $A \in \mathcal{F}_x$ iff $x \in A$. Such ultrafilters are called **principal**; they correspond to maximal ideals $\mathfrak{m}_x$, $x \in I$.

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Ultraproducts of sets and algebraic structures

Let $\mathcal{F}$ be a non-principal ultrafilter on a set I, and let $\{X_i\}_{i\in I}$ be a collection of sets.

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is the set of functions

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defined on some domain $A_x \in \mathcal{F}$, modulo the equivalence relation

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Thus, unlike the usual product, the ultraproduct can be non-empty even if some of the factors are empty.

First-order structures and first-order properties are compatible with this construction. For example, ultraproducts of groups, rings, modules, and fields are again groups, rings, modules, and fields.

Ultraproducts of sets and algebraic structures

More precisely, a property P is said to hold **for $\mathcal{F}$-almost all i** if

$$\{\, i \in I \mid X_i \text{ satisfies } P \,\} \in \mathcal{F}.$$

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Also K has cardinality continuum. Hence, by Steinitz's theorem,

$$K \cong \mathbb{C}.$$

Ultraproducts of sets and algebraic structures

As another example, let I be the set of primes and consider

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What is its characteristic? For every fixed prime ℓ, the characteristic is not ℓ, since $\operatorname{char} \overline{\mathbb{F}}_p \neq \ell$ for $\mathcal{F}$-almost all primes p (in fact, all but one). Hence

$$\operatorname{char} L = 0.$$

Therefore, again by Steinitz's theorem,

$$L \cong \mathbb{C}.$$

Lecture 3

Categorical ultraproducts

Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$, and let $\mathcal{C}_n$ be essentially small K_n-linear categories. One can form the **categorical ultraproduct**

$$\mathcal{U} := \prod_{n \in \mathbb{N}}^{\mathcal{F}} \mathcal{C}_n .$$

Its objects are sequences $X = (X_n)$, $X_n \in \mathcal{C}_n$ defined for n in some domain $A_X \in \mathcal{F}$, and its Hom spaces are

$$\mathrm{Hom}_{\mathcal{U}}(X, Y) = \prod_{n \in \mathbb{N}}^{\mathcal{F}} \mathrm{Hom}_{\mathcal{C}_n}(X_n, Y_n),$$

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If the $\mathcal{C}_n$ are rigid symmetric monoidal, then so is $\mathcal{U}$, with tensor product, duality, associativity, and symmetry defined componentwise.

However, $\mathcal{U}$ need not be a symmetric tensor category. The condition that every object have finite length is not first-order. Similarly, finite dimensionality of Hom spaces need not be preserved by the ultraproduct.

The generated subcategory has finite length

Now take

$$\mathcal{U} = \prod_{n\in\mathbb{N}}^{\mathcal{F}} \mathrm{Rep}_{\overline{\mathbb{Q}}} GL_n, \qquad V = (\overline{\mathbb{Q}}, \overline{\mathbb{Q}}^2, \overline{\mathbb{Q}}^3, \ldots).$$

Although $\mathcal{U}$ itself may fail the artinian condition, we restrict to the full tensor subcategory

$$\langle V \rangle \subset \mathcal{U}$$

generated by V: its objects are subquotients of finite direct sums of

$$V^{\otimes r} \otimes V^{*\otimes s}, \qquad r, s \geq 0.$$

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Hence their dimensions stabilize. Consequently the Hom spaces in $\langle V\rangle$ are finite-dimensional, and all objects of $\langle V\rangle$ have finite length. Thus $\langle V\rangle$ is a symmetric tensor category.

The dimension parameter in the ultraproduct

The categorical dimension of V is the ultraproduct of the ordinary dimensions:

$$\dim(V) = [1, 2, 3, \ldots] \in K = \prod_{n \in \mathbb{N}}^{\mathcal{F}} \overline{\mathbb{Q}} \cong \mathbb{C}.$$

Choose a field isomorphism

$$\xi : K \xrightarrow{\sim} \mathbb{C}.$$

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Deligne's theorem states that

$$\langle V \rangle \simeq \operatorname{Rep} GL_t$$

for this transcendental value of t. Indeed, by the universal property of $\operatorname{Rep} GL_t$, there is a unique symmetric tensor functor $H : \operatorname{Rep} GL_t \to \langle V \rangle$ with $H([1, 0]) = V$, and it is easy to check that this functor is an equivalence.

Dependence on the field isomorphism

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If $t' \in \mathbb{C}$ is another transcendental number, an isomorphism

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Thus, for a fixed non-principal ultrafilter, the same categorical ultraproduct realizes

$$\operatorname{Rep} GL_t$$

for every transcendental t, after a suitable identification of the ultraproduct field with $\mathbb{C}$.

Algebraic parameters by reduction modulo primes

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Proof. For large N, there are roughly $N^{1/\deg(q)}$ positive integers of the form $q(n)$ with $q(n) < N$, whereas for a fixed collection of m primes there are on the order of $(\log N)^m$ positive integers less than N divisible only by those primes.

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Denote this infinite set of primes by $\mathbb{A}_q$. For $p \in \mathbb{A}_q$ and choose a root $\overline{n_p} \in \mathbb{F}_p$ of q represented by a positive integer n_p. Necessarily $n_p \to \infty$.

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Recall that the field ultraproduct $K = \prod_{p \in \mathbb{A}_q}^{\mathcal{F}} \overline{\mathbb{F}}_p$ may be identified with $\mathbb{C}$. Under a suitable identification, the element $[n_p] \in K$ maps to the prescribed root t of q.

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With $V = (\overline{\mathbb{F}}_p^{\,n_p})_p$, from the universal property of $\mathrm{Rep}\, GL_t$ one obtains

$$\mathrm{Rep}\, GL_t \simeq \langle V \rangle \subset \prod_{p \in \mathbb{A}_q}^{\mathcal{F}} \mathrm{Rep}_{\overline{\mathbb{F}}_p}(GL_{n_p}).$$

Integral parameters and abelian envelopes

The case $t \in \mathbb{Z}$ (so $q(x) = x - t$, $\overline{n_p} = t$) is the same, except $n_p \to \infty$ is not automatic and needs to be arranged. For example, one may take $n_p = p + t$.

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Fixed positive characteristic

One may also use the ultrafilter approach to construct symmetric tensor categories of non-moderate growth in positive characteristic.

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One may also use the ultrafilter approach to construct symmetric tensor categories of non-moderate growth in positive characteristic. To this end, fix a prime p and work over algebraically closed fields of characteristic p. Consider

$$\mathcal{U}_p = \prod_{n\in\mathbb{N}}^{\mathcal{F}} \mathrm{Rep}_{\overline{\mathbb{F}}_p}(GL_n), \qquad V = (\overline{\mathbb{F}}_p, \overline{\mathbb{F}}_p^{\,2}, \overline{\mathbb{F}}_p^{\,3}, \ldots).$$

As before, the subcategory $\langle V\rangle \subset \mathcal{U}_p$ is a symmetric tensor category over the ultraproduct field.

Fixed positive characteristic

One may also use the ultrafilter approach to construct symmetric tensor categories of non-moderate growth in positive characteristic. To this end, fix a prime p and work over algebraically closed fields of characteristic p. Consider

$$\mathcal{U}_p = \prod_{n\in\mathbb{N}}^{\mathcal{F}} \mathrm{Rep}_{\overline{\mathbb{F}}_p}(GL_n), \qquad V = (\overline{\mathbb{F}}_p, \overline{\mathbb{F}}_p^{\,2}, \overline{\mathbb{F}}_p^{\,3}, \ldots).$$

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But now we encounter a new phenomenon. We have $\dim(V) = [1, 2, 3, \ldots] \in K = \prod_n^{\mathcal{F}} \overline{\mathbb{F}}_p$. However, this is not an arbitrary element.

Why dimensions lie in $\mathbb{F}_p$

Lemma. Let $\mathcal{C}$ be a symmetric tensor category over a field of characteristic $p > 0$, and let $X \in \mathcal{C}$ have categorical dimension d. Then $d \in \mathbb{F}_p$.

Why dimensions lie in $\mathbb{F}_p$

Lemma. Let $\mathcal{C}$ be a symmetric tensor category over a field of characteristic $p > 0$, and let $X \in \mathcal{C}$ have categorical dimension d. Then $d \in \mathbb{F}_p$.

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Exterior powers and p-adic dimension

The missing information is contained in the dimensions of exterior powers. For $X \in \mathcal{C}$, set

$$t_i := \dim\bigl(\wedge^{p^i} X\bigr) \in \mathbb{F}_p, \qquad i \geq 0.$$

These digits define the p-**adic dimension**

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Equivalently, one can show that $\mathrm{Dim}(X)$ is characterized by the generating function

$$\sum_{m \geq 0} \dim(\wedge^m X) z^m = (1+z)^{\mathrm{Dim}(X)},$$

where, for $t = \cdots t_2 t_1 t_0$,

$$(1+z)^t = (1+z)^{t_0} (1+z^p)^{t_1} (1+z^{p^2})^{t_2} \cdots .$$

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Recall that if X is a compact metric space and $x_1, x_2, \ldots$ is a sequence in X, then by the **Bolzano-Weierstrass theorem** it has a convergent subsequence.

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It turns out that $t = \lim_{n\to\infty}^{\mathcal{F}} n$. Moreover, a good news is that the resulting category $\langle V \rangle$ depends only on t, not on the remaining nonconstructive choices. We denote his category by

$$\mathrm{Rep}^{ab}\, GL_t, \qquad t \in \mathbb{Z}_p.$$

Thus in characteristic p, interpolation of rank is naturally p-adic.

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The category $\mathrm{Rep}_{\Bbbk}(\mathbb{Z}/p)$

Let $\operatorname{char}\Bbbk = p$. Then

$$\Bbbk[\mathbb{Z}/p] \cong \Bbbk[g]/(g^p - 1) = \Bbbk[g]/(g-1)^p.$$

Therefore the indecomposable $\Bbbk[\mathbb{Z}/p]$-modules are

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The categorical dimension of J_n is $n \in \Bbbk$. Hence

$$\dim(J_p) = p = 0.$$

By the criterion from Lecture 2, the dimension-zero indecomposable J_p becomes zero after semisimplification, while $J_1, \ldots, J_{p-1}$ give the simple objects.

Definition of Ver_p

The **Verlinde category** is the semisimplification

$$\mathrm{Ver}_p := \mathrm{Rep}_{\Bbbk}(\mathbb{Z}/p)^{ss}.$$

Let

$$S : \mathrm{Rep}_{\Bbbk}(\mathbb{Z}/p) \to \mathrm{Ver}_p$$

be the quotient functor. The simple objects of Ver_p are

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It is the basic positive-characteristic replacement for sVec.

The Verlinde fusion rule

The tensor product in Ver_p is the truncated Clebsch–Gordan rule:

$$L_m \otimes L_n = \bigoplus_{i=1}^{\min(m,n,p-m,p-n)} L_{|m-n|+2i-1}.$$

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The same fusion rules occur for $\widehat{\mathfrak{sl}}_2$ at level $p-2$ (**Verlinde algebra** in the **2-dimensional WZW conformal field theory**), which explains the name “Verlinde category".

Small primes and the first exotic example

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If a fiber functor $\mathrm{Ver}_5 \to \mathrm{sVec}_{\Bbbk}$ existed, the integer superdimension d of the image of L_3 would satisfy

$$d^2 = 1 + d.$$

This equation has no integral solution. Hence Ver_5 is not super-Tannakian.

Moderate growth of Ver_p

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More generally, a similar argument shows that for all $p \geq 5$, Ver_p is a symmetric tensor category of moderate growth which is not super-Tannakian. This is the simplest way in which Deligne's characteristic-zero theorem fails in characteristic p.

The copy of sVec inside Ver_p

For $p > 2$, the last simple object satisfies

$$L_{p-1} \otimes L_{p-1} \cong L_1, \qquad \dim(L_{p-1}) = p - 1 = -1.$$

Thus L_1 and L_{p-1} generate a copy of $\mathrm{sVec}_{\Bbbk}$ inside Ver_p.

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Let Ver_p^+ be the tensor subcategory generated by the odd-indexed simple objects:

$$L_1, L_3, L_5, \ldots .$$

Then, for $p > 2$,

$$\mathrm{Ver}_p \simeq \mathrm{Ver}_p^+ \boxtimes \mathrm{sVec}_{\Bbbk} .$$

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Thus Ver_p contains $\mathrm{sVec}_{\Bbbk}$, but it is strictly larger when $p \geq 5$.

Other constructions of Ver_p

The same symmetric tensor category arises in several equivalent ways.

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These realizations connect Ver_p with modular Lie theory, tilting modules, and quantum groups.

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Thus if we want to generalize Deligne's theorem to characteristic p, the target of fiber functors must be enlarged to Ver_p.

The positive-characteristic Deligne theorem, semisimple case

In fact, for semisimple symmetric tensor categories such a generalization does hold!

Theorem (Ostrik; Coulembier–Etingof–Ostrik)

Let $\mathcal{C}$ *be a semisimple symmetric tensor category of moderate growth over an algebraically closed field of characteristic* $p > 0$*. Then* $\mathcal{C}$ *admits a unique up to isomorphism fiber functor*

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Ostrik proved the theorem for fusion categories. The general version was later proved by Coulembier–Etingof–Ostrik.

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The correct target of fiber functors for non-semisimple categories therefore requires larger target categories than Ver_p.

Prime-power Verlinde categories

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It also arises from tilting modules for SL_2 in characteristic p, after quotienting by the tensor ideal generated by the n-th Steinberg module.

Prime-power Verlinde categories

There is a nested sequence of incompressible symmetric tensor categories in characteristic p:

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Let

$$\mathrm{Ver}_{p^\infty} := \bigcup_{n \geq 1} \mathrm{Ver}_{p^n} .$$

The conjectural general theorem

Conjecture (Benson–Etingof–Ostrik)

Let $\mathcal{C}$ *be a symmetric tensor category of moderate growth over an algebraically closed field of characteristic* $p > 0$*. Then* $\mathcal{C}$ *admits a unique fiber functor*

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This would be the full positive-characteristic analogue of Deligne's theorem.

The semisimple theorem says that, in the semisimple case, the target Ver_{p^∞} can be replaced by the first term Ver_p.

The Frobenius functor

Let $\mathcal{C}$ be a symmetric tensor category in characteristic p. For $X \in \mathcal{C}$, the cyclic group $\mathbb{Z}/p$ acts on $X^{\otimes p}$ by cyclic permutation.

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Thus $X^{\otimes p}$ is an object of

$$\mathcal{C} \boxtimes \mathrm{Rep}(\mathbb{Z}/p).$$

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A key point is that Fr is additive, although $X \mapsto X^{\otimes p}$ is not. The non-additive part is killed by semisimplification.

Frobenius exactness

A symmetric tensor category $\mathcal{C}$ in characteristic p is called **Frobenius exact** if the Frobenius functor

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The categories Ver_{p^n} for $n \geq 2$ are not Frobenius exact; this is why they lie beyond the Ver_p-valued theorem.

The Frobenius-exact theorem

Theorem (Coulembier–Etingof–Ostrik)

Let $\mathcal{C}$ be a symmetric tensor category of moderate growth over an algebraically closed field of characteristic $p > 0$. Then $\mathcal{C}$ is Frobenius exact if and only if it admits a fiber functor

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This extends the semisimple theorem to a natural non-semisimple class.

It also identifies the precise obstruction to having Ver_p as the target: failure of exactness of the Frobenius functor.

Local semisimplicity criterion

Frobenius exactness has a useful intrinsic characterization.

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It is equivalent to the existence of an abelian $\mathbb{k}$-linear symmetric monoidal category $\mathcal{D}$, not necessarily a tensor category, and an exact symmetric monoidal functor

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Equivalently, for filtered objects, symmetric powers commute with passing to the associated graded:

$$\mathrm{Sym}^\bullet(\mathrm{gr}\, X) \xrightarrow{\sim} \mathrm{gr}\big(\mathrm{Sym}^\bullet X\big).$$

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the **Frobenius-Perron dimension** of $F(X)$, which is the largest eigenvalue of the matrix of multiplication by $F(X)$ in the Verlinde fusion ring. Thus the asymptotic growth of tensor powers in $\mathcal{C}$ is computed inside the explicit fusion category Ver_p.

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This is one of the bridges from positive-characteristic tensor categories to modular representation theory.

Lecture 4

Modular representations and the non-negligible part

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The tensor powers $V^{\otimes n}$ may have many summands which are killed by semisimplification. Let $\overline{V}$ denote the image of V in the semisimplification of $\mathrm{Rep}(G)$, and define

$$\overline{d}_n(V) := \text{the number of indecomposable non-negligible summands of } V^{\otimes n},$$

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Equivalently,

$$\overline{d}_n(V) = \mathrm{length}(\overline{V}^{\otimes n}).$$

Consider the asymptotic invariant

$$\delta(V) := \lim_{n\to\infty} \overline{d}_n(V)^{1/n} = \mathrm{gd}(\overline{V})$$

(it exists by the Fekete lemma since $\overline{d}_{n+m}(V) \geq \overline{d}_m(V)\overline{d}_n(V)$).

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(it exists by the Fekete lemma since $\overline{d}_{n+m}(V) \geq \overline{d}_m(V)\overline{d}_n(V)$).

Thus $\delta(V)$ measures the exponential growth rate of the part of $V^{\otimes n}$ which survives in the semisimplified tensor category.

How the Etingof–Coulembier–Ostrik theorem enters

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For the simple objects $L_1, \ldots, L_{p-1}$ of Ver_p, write

$$F(\overline{V}) = \bigoplus_{k=1}^{p-1} m_k L_k, \qquad m_k \in \mathbb{Z}_{\geq 0}.$$

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In Ver_p,

$$\mathrm{FPdim}(L_k) = [k]_q = \frac{\sin(\pi k/p)}{\sin(\pi/p)}.$$

Therefore

$$\delta(V) = \mathrm{gd}(\overline{V}) = \sum_{k=1}^{p-1} m_k [k]_q.$$

The numerical form

The preceding reduction turns a modular representation-theoretic problem into arithmetic in the Verlinde fusion ring.

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Theorem (Etingof–Coulembier–Ostrik, applied to $\mathrm{Rep}(G)$)

Let $V \in \mathrm{Rep}(G)$*, and define* $\delta(V)$ *as above. Then there exist unique non-negative integers* $m_1, \ldots, m_{p-1}$*, characterized by the following Verlinde numerical data:*

$$\delta(V) = \sum_{k=1}^{p-1} m_k [k]_q .$$

If $p > 2$*, the same multiplicities also satisfy*

$$\delta(S^2 V) - \delta(\Lambda^2 V) = \sum_{k=1}^{p-1} m_k [k]_{q^2} .$$

Congruences and bounds

With $F(\overline{V}) = \oplus_k m_k L_k$, the theorem also gives the congruence

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If $|G|$ is divisible by p and V is faithful, then

$$\delta(V) < \dim_{\Bbbk} V.$$

If $\dim V = d \neq 0$ in $\mathbb{F}_p$, $1 \leq d \leq p-1$, then

$$\delta(V) \geq [d]_q .$$

What the theorem says in small ranks

The theorem is especially effective when $\dim V$ is small compared with p.

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For $p = 2$ or $p = 3$, all Frobenius-Perron dimensions of simples in Ver_p are equal to 1. Hence $\delta(V)$ is an integer.

For $p = 5$,

$$[1]_q = [4]_q = 1, \qquad [2]_q = [3]_q = \frac{1+\sqrt{5}}{2}.$$

Thus

$$\delta(V) = a + b\frac{1+\sqrt{5}}{2}, \qquad a, b \in \mathbb{Z}_{\geq 0}.$$

Interpretation for tensor powers of modules

The full tensor power $V^{\otimes n}$ has dimension $(\dim V)^n$, but the non-negligible part has growth rate $\delta(V)^n$.

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The theorem therefore converts questions about indecomposable summands of $V^{\otimes n}$ into the finite calculation of the image of $\overline{V}$ in the fusion ring $K_0(\mathrm{Ver}_p)$.

Beyond moderate growth

The preceding results depend on moderate growth and on the existence of a fiber functor to Ver_p.

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There is another source of symmetric tensor categories, developed by Harman–Snowden, which deliberately moves outside this moderate-growth framework.

The idea is to replace a sequence of finite permutation groups by one infinite permutation group, but to retain finite-dimensional Hom spaces by imposing a finite-orbit condition.

This leads to tensor categories attached to **oligomorphic groups** equipped with a measure. In the case of the infinite symmetric group one recovers Deligne's $\mathrm{Rep}(S_t)$; for other groups one obtains genuinely new pre-Tannakian categories.

Oligomorphic groups

Let a group G act faithfully on a set Ω.

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The natural topology on G has as a neighborhood basis of the identity the pointwise stabilizers of finite subsets of Ω. A G-set is called **smooth** if all stabilizers are open; all G-sets we consider will be assumed smooth. A G-set is **finitary** if it has finitely many G-orbits.

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Example: $S_\infty = \mathrm{Perm}(\mathbb{N})$ acting on $\Omega = \mathbb{N}$ is oligomorphic, since orbits on Ω^n are described by equality patterns.

Measures as generalized indices

A **measure** is the substitute for the cardinality of a finite orbit.

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Fix a commutative coefficient ring k. A k-valued measure μ assigns to each finitary $\widehat{G}$-set X (up to isomorphism) an element $\mu(X) \in k$, subject to the following rules:

$$\mu(\text{point}) = 1, \qquad \mu(X \sqcup Y) = \mu(X) + \mu(Y), \qquad \mu(X \times Y) = \mu(X)\mu(Y),$$

with multiplicativity for transitive fibrations: If X, Y are transitive G-sets and $f : X \to Y$ is a G-equivariant map, then

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for $y \in Y$. Equivalently, μ is a consistent system of generalized indices for open subgroups. For S_∞, the measure of the basic permutation set Ω is the Deligne parameter t. Harman–Snowden package all measures into a universal ring $\Theta(G)$, which has a presentation arising from the above axioms. Then choosing a measure amounts to choosing a specialization homomorphism $\Theta(G) \to k$.

Given a measure μ, we can integrate a U-invariant k-valued function f on a finitary U-set Y. Namely,

$$\int_Y f \, d\mu = \sum_{z \in k} z \cdot \mu(f^{-1}(z)).$$

(this sum has finitely many nonzero terms).

The permutation tensor category

Given (G, μ), Harman–Snowden define a rigid pseudotensor category

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For every finitary G-set X there is an object Vec_X. Direct sum and tensor product are determined by the corresponding set operations:

$$\mathrm{Vec}_X \oplus \mathrm{Vec}_Y = \mathrm{Vec}_{X \sqcup Y}, \qquad \mathrm{Vec}_X \otimes \mathrm{Vec}_Y = \mathrm{Vec}_{X \times Y}.$$

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Thus $\mathrm{Perm}(G; \mu)$ is a linearized category of finitary smooth G-sets: disjoint union becomes direct sum, cartesian product becomes tensor product, and generalized cardinality enters through μ.

The abelian envelope

The category Perm$(G;\mu)$ is typically not abelian. Harman–Snowden therefore construct a completed group algebra

$$A_k(G;\mu) = \varprojlim_{U} C(G/U)$$

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Suppose k is a field. Harman and Snowden showed that if $\mathsf{Perm}(G;\mu)$ has trace zero property and μ is quasi-regular then $\mathsf{Rep}(G;\mu)$ is a symmetric tensor category which is an abelian envelope of $\mathsf{Perm}(G;\mu)$; moreover, in the regular case it is semisimple.

Basic examples

For $G = S_\infty$, orbits of G on Ω^n are labeled by set partitions of $[1, n]$ corresponding to equality patterns. A complex-valued measure is determined by the condition

$$\mu(\Omega) = t \in \mathbb{C}.$$

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For finite linear groups in the stable limit, the same method gives analogues related to Knop's interpolation categories $\operatorname{Rep} GL_t(\mathbb{F}_q)$.

The Delannoy category

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Harman and Snowden show that this group has exactly **four** measures with values in any nonzero ring. Only one of them is (quasi)regular. This measure μ is defined by the rule that for an orbit $X \subset \mathbb{R}^n$, $\mu(X)$ is the Euler characteristic of X with compact supports; in particular, $\mu(\mathbb{R}) = -1$.

For this measure, the Harman-Snowden construction gives a symmetric tensor category called the **Delannoy category**, the simplest genuinely new example in this framework.

The Delannoy category

Let $G = \mathrm{Aut}(\mathbb{R}, <)$. For $n \geq 0$, set

$$\mathbb{R}^{(n)} := \{(x_1, \dots, x_n) : x_1 < \cdots < x_n\}.$$

These are the transitive finitary G-sets.

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The **Delannoy category** is the semisimple symmetric tensor category obtained from the Harman–Snowden category attached to this measured oligomorphic group:

$$\mathcal{D} := \mathsf{Rep}(\mathrm{Aut}(\mathbb{R}, <); \mu).$$

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Equivalently, one starts with the permutation objects

$$C(\mathbb{R}^{(n)}), \qquad n \geq 0,$$

and passes to the Karoubian/abelian envelope supplied by the general theory.

Delannoy paths and Hom spaces

An (n, m)-**Delannoy path** is a lattice path from $(0, 0)$ to (n, m) using the three steps

$$(1, 0), \qquad (0, 1), \qquad (1, 1).$$

Let $D(n, m)$ be the number of such paths, called the **Delannoy numbers**. They were studied by **Henri-Auguste Delannoy** (1833 -1915), a French army officer and amateur mathematician.

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The reason for the name “Delannoy category" is that G-orbits on

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Consequently,

$$\dim \operatorname{Hom}\big(C(\mathbb{R}^{(m)}), C(\mathbb{R}^{(n)})\big) = D(n, m).$$

Composition is convolution of these kernels, with the Euler measure providing the signs and finite values of the relevant infinite sums.

Structure of the Delannoy category

The Delannoy category has a completely explicit description.

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Tensor products are described by shuffling words, but with collisions allowed. This "shuffle with collisions" is the categorical shadow of the diagonal step $(1, 1)$ in a Delannoy path.

THANK YOU!